Lecture Notes

Constrained Quantization: Summer Training Activities CBPF.

C.H. Lima[a,b,1]

[a] *IFT-UNESP, Brazil*

E-mail: carloscdlima@gmail.com

Abstract: These are lectures notes from the mini-course given in the Summer Training Activities of CBPF(Brazilian Center for Research in Physics) at Rio de Janeiro from 02/19/2019 to 02/27/2019 about Constrained Quantization. There will be a focus on the Dirac Program of canonical quantization as it is the basis for the more modern approaches such as BRST and BV quantization. To do that it will be cover how to deal with constraints in classical physics and the challenges of making it quantum. Trough the discussion it is reviewed the problem of canonical quantization and how to implement constraint in the quantum level. In the lecture is presented alternative approaches for constrained quantization but only at an introductory level, focusing more on the problem of gauge like constraints.

Contents

1 Lecture I.

1.1 Introduction

Constrained Dynamics are all around Physics, from classical systems like describing the trajectory of a pendulum or more complicated systems like electromagnetism or even gravity and its quantum counterpart. During our formation in Physics, especially after doing a course in quantum field theory it is expected to be familiar with the concept of gauge transformation. This kind of transformation appear first in Classical Electromagnetism but stays with us until the description of the fundamental interactions. These kinds of theories are actually constrained systems as we will see in more detail. The need for constrained systems occur a lot in Physics because the necessity of using more coordinates than it is needed to describe a system. We can see a example of this in the case of a pendulum. It is possible to use x and y coordinates to describe its movement but ultimately the system is one dimensional and it is well described by the coordinate θ. In the x and y coordinate we have the constraint:

$$x^2 + y^2 = r^2 \tag{1.1}$$

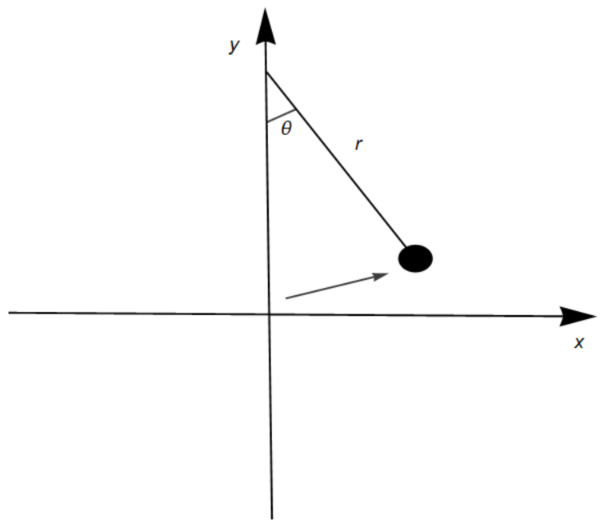

Figure 1. One dimensional pendulum described by both coordinates systems

The constraint appear in this system to kill the additional degree of freedom that we putted by using more coordinates than degrees of freedom. In this case we can freely choose to work with (x, y) or θ coordinates. However, there are cases where one is forced to use more coordinates because we want a given symmetry to be manifest, like in electromagnetism that one has 2 degrees of freedom but we use a vector potential with 4 components. This is the case of 4D gravity as well that we know has 2 degrees of freedom but we use a metric that has 10 independent components. Constraints in general help us to deal with these extra degrees of freedom, but we need to be careful when dealing with such a system. Another great motivation to study constrained dynamics appear when one goes to relativistic systems. In the description of a relativistic particle we need to enforce the mass shell condition:

$$p^2 + m^2 = 0 \tag{1.2}$$

This states that not all momentum p_μ are independent, this is indeed a constraint in the Phase Space. Learn how to deal with such systems is of great importance, when going to a quantum description the problems became even more difficult so a good understanding of the subject is necessary to move ahead. We will see how to categorize the different types of constrained systems and how to deal with them classically. The main goal of these lectures is to understand classical constrained systems and how to move this kind of system to the quantum realm. It will be discussed the problems of the usual quantum map and the ambiguities that arise from that. After that we will work out some examples of quantum constrained systems, finishing with the usual electromagnetism. The direction that I will follow is sometimes called Dirac Quantization program or in the case of string theory, Old Covariant Quantization. As the name suggests it is an old method that has all the fundamental ingredients to treat quantum mechanically a constrained system, but in the examples will become clear that such treatment is not always straightforward. This motivated the creation of more powerful tools to quantize constrained systems, such as BRST or BV Quantization. One could ask why study this method if there are more efficient ways to do the same thing? A simple answer is that all methods rely on understanding the Dirac Program first. In some sense, they only automatize the method by adding additional objects to the theory (ghosts). Now that the goal is set let's see how this lectures will be arranged:

- **First Lecture:** Introduction of the subject, review of Hamiltonian dynamics, an introduction of classical constraints, Lagrange multipliers and classification of constraints.
- **Second Lecture:** How to deal with classical constraints.
- **Third Lecture:** Review of Quantization of non-singular theories, Quantization of second class constraint and Quantization of first class constraints(Dirac Program)
- **Fourth Lecture:** Dirac Quantization of the non-relativistic and relativistic Free Particle.
- **Fifth Lecture:** Dirac Quantization of the Closed Bosonic String.
- **Sixth Lecture:** Dirac Quantization of the Electromagnetic Field, General remarks and the road ahead.

The bibliography used to write this lecture notes and recommended about this subject is:

1. Quantization of Gauge Systems from Marc Henneaux & Claudio Teitelboim.

2. Lectures on Quantum Mechanics from Paul Dirac.

3. Constrained Dynamics from Kurt Sundermeyer.

4. Quantization of Fields with Constraints from D. M. Gitman & Igor V. Tyutin

With that settle down we can start the review on classical mechanics that will be useful to understand constrained dynamics, it will be worked only finite dimensional systems. The generalization to infinite dimensional systems is straightforward in this case.

1.2 Things that probably you already know.

The starting point for us is to assume a action functional $\mathcal{S}$ that generate the equation of motion of a given system:

$$\mathcal{S} = \int \mathrm{d}t \, L \tag{1.3}$$

The system is described by n local coordinates $\{q^i\}$. The Lagrangian L is normally written in terms of a kinetic and a potential part:

$$L = T - V \tag{1.4}$$

In general L can be any function of $(q, \dot{q}, t)$[1] that generate a valid equation of motion under extremization of such action. Given such a action we can find its equation of motion by finding its extreme:

$$\delta\mathcal{S} = 0 \tag{1.5}$$

This gives us the usual Euler-Lagrange equations under the assumption that q is fixed in the end points:

$$\delta\mathcal{S} = \int \mathrm{d}t \, \delta L(q, \dot{q}, t) = \int \mathrm{d}t \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \right) \tag{1.6}$$

Doing a integration by parts and dropping out the boundary term we get:

$$\int \mathrm{d}t \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \right) \delta q^i = 0 \tag{1.7}$$

Because δq^i is arbitrary and independent we get the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1.8}$$

These are N second order differential equations that by solving it under appropriate boundary conditions fixes our system. The good thing of using Lagrangian to describe our system is that most of the symmetries are usually manifest in this formalism, for

[1] We will work with time independent Lagrangian because it has a better geometrical interpretation when going to the Hamiltonian description.

instance, Lorentz symmetry in relativistic systems. The problem in quantizing directly in such formalism is that one would need to rely on path integral quantization. This can be avoided if we move into Hamiltonian description of the system, where the Hamiltonian has a clear physical interpretation and can be used to quantize a theory easily. In a non-relativistic theory, this generates the usual Schrodinger equation for the wave function. Because we want to do a canonical quantization in some sense we will need the Hamiltonian description of the system. So let's describe the path from Lagrangian to Hamiltonian mechanics. The heart of this process is to trade n second order differential equations for $2N$ first order differential equations. The first step is to define the canonical momentum $\{p_i\}$, such that we can do a Legendre transformation using them(trade all the $\dot{q}$ dependence for this momentum p):

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{1.9}$$

This choice can be justified by looking for the total differential of the Lagrangian:

$$\mathrm{d}L = \frac{\partial L}{\partial q^i}\mathrm{d}q^i + \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}\dot{q}^i + \frac{\partial L}{\partial t}\mathrm{d}t \tag{1.10}$$

One can re-write this as:

$$\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\mathrm{d}\dot{q}^i - L\right) = -\frac{\partial L}{\partial t}\mathrm{d}t - \frac{\partial L}{\partial q^i}\mathrm{d}q^i + \dot{q}^i\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \tag{1.11}$$

Now we define the Hamiltonian as:

$$H = \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}\dot{q}^i - L \tag{1.12}$$

This give us the total differential of the Hamiltonian H:

$$\mathrm{d}H = -\frac{\partial L}{\partial t}\mathrm{d}t - \frac{\partial L}{\partial q^i}\mathrm{d}q^i + \dot{q}^i\mathrm{d}p_i \tag{1.13}$$

The interesting thing now is that we can interpret the Hamiltonian as a function of (q^i, p_i, t) instead of the usual $(q^i, \dot{q}^i, t)$ in the Lagrangian description. If we can invert all velocities in terms of canonical momentum this map is straightforward, we call such systems non-singular. The Hamilton equations of motion are:

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{1.14}$$

$$\frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i} \tag{1.15}$$

$$\frac{\partial H}{\partial p_i} = \dot{q}^i \tag{1.16}$$

Using the Euler-Lagrange equations on the (1.15) we can write the Hamilton equations in a more familiar form:

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{1.17}$$

$$\frac{\partial H}{\partial q^i} = -\dot{p}_i \tag{1.18}$$

$$\frac{\partial H}{\partial p_i} = \dot{q}^i \tag{1.19}$$

For what will follow we need to introduce another important object that plays a major role in the Hamiltonian description, the Poisson Bracket between two phase space functions:

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i} \tag{1.20}$$

This is a explicit form for the Poisson bracket in a specific set of coordinates (Darboux Coordinates)[2]. This Poisson bracket has some nice proprieties:

- Antisymmetry: $\{F, G\} = -\{G, F\}$.
- Linearity(In both slots): For z_1,z_2 constants $\in \mathbb{C}$, $\{z_1F_1 + z_2F_2, G\} = z_1\{F_1, G\} + z_2\{F_2, G\}$.
- Null element: For z_1 constant, $\{z_1, F_1\} = 0$.
- Jacobi identity: $\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0$.
- Leibniz's rule(In both slots): $\{A, BC\} = \{A, B\}C + B\{A, C\}$

Using the Poisson bracket we can write the time evolution as:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \{\_, H\} + \frac{\partial}{\partial t} \tag{1.21}$$

So the equations of motions are:

$$\dot{q}^i = \{q^i, H\} \tag{1.22}$$

$$\dot{p}_i = \{p_i, H\} \tag{1.23}$$

So if we derive a Hamiltonian we can find its equations of motion and then solve it uniquely given a set of boundary conditions. The fun begins when we try to treat

[2]Transformations that preserves this form are called canonical transformations and plays a important role in classical physics.

constrained systems, we saw that for the Hamiltonian to exist and be unique one need to invert the momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{1.24}$$

Such that:

$$\dot{q}^i = \dot{q}^i(q, p, t) \tag{1.25}$$

However there is a catch, only in regular systems one can solve uniquely for $\dot{q}^i$. The restriction that demands (1.25) to be true is:

$$\det\left(\frac{\partial p_i}{\partial \dot{q}^j}\right) = \det\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right) = \det(R_{ij}) \neq 0 \tag{1.26}$$

If this holds it is possible to write a unique solution for $\dot{q}^i = \dot{q}^i(q, p, t)$. Then what follows is that the map from Lagrangian and Hamiltonian formulation has no obstruction. Meanwhile, if this determinant is zero then the system is singular and this means that we have a constraint. Not all the coordinates and velocities are independent, it exists a relation between them at any instance of time:

$$C(q^i, \dot{q}^i, t) = 0 \tag{1.27}$$

It is important to point out that the singular characteristic is independent of coordinate choice and can't be gone by a redefinition of the Lagrangian by a total derivative. This matrix (1.26) appears in the equation of motion multiplying the acceleration, this means that in a singular theory the acceleration is not uniquely determined by the position and velocities and the solutions may have arbitrary functions of time as we will see. To see the relation between singular theories we can write the total time derivative in the Euler Lagrange equation as:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{q}^j \frac{\partial}{\partial q^j} + \ddot{q}^j \frac{\partial}{\partial \dot{q}^j} + \frac{\partial}{\partial t} \tag{1.28}$$

Then It is easy to cast in the form:

$$\ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} = \frac{\partial L}{\partial q^i} - \dot{q}^j \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} - \frac{\partial^2 L}{\partial t \partial \dot{q}^i} = U_i(q, \dot{q}, t) \tag{1.29}$$

If the determinant of R_{ij} is zero, exists at least one zero mode with eigenvector ξ^i[3]. Contracting the Euler Lagrange equations with these eigenvector we obtain:

$$\xi_i U_i(q, \dot{q}) = C(q^i, \dot{q}^i, t) = 0 \tag{1.30}$$

This is a constraint, it is a relation between the position and velocities that are true in every instant of time. The constraint here is appearing in the Lagrangian formalism, to

[3]The determinant is the product of the eigenvalues so at least one eigenvalue is zero and we would have $R_{ij}\xi^j = 0$.

have a nice geometrical interpretation and the easy possibility to quantize the system let's see how this appears in the Hamiltonian formalism (Let's work only with time independent systems). In there the constraints relate momentum and coordinates as:

$$C(q^i, p_i) = 0 \tag{1.31}$$

This means that the Hamiltonian is not uniquely determined, before we see how to deal with this there is a additional thing that need to be noticed. Because of the proprieties of the Hamiltonian dynamics any phase space function can be treated like a "Hamiltonian"[4], having a associated vector field and integral lines (Giving a preferred direction on phase space):

$$\frac{dq^i}{d\lambda_f} = \{q^i, f\} \tag{1.32}$$

$$\frac{dp_i}{d\lambda_f} = \{p_i, f\} \tag{1.33}$$

For a conserved quantity I we can do the same:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0 = \{H, I\} = -\{I, H\} \tag{1.34}$$

So it is clear to see that a constraint is a conserved quantity in some sense, as is valid at all times. The same argument tells us that they define a symmetry in phase space and generate a corresponding flow as we will see.

The next step is to find a way to use the Hamiltonian formalism even with these arbitrariness appearing because of the singular nature of the constraint. If we want that we need to add additional objects called Lagrangian Multipliers. To see the need for this we can look for a simple constrained Lagrangian:

$$L = \frac{1}{2}(\dot{q}_1 + \dot{q}_2)^2 \tag{1.35}$$

The momenta are:

$$p_1 = \dot{q}_1 + \dot{q}_2 \tag{1.36}$$

$$p_2 = \dot{q}_1 + \dot{q}_2 \tag{1.37}$$

This system has a constraint:

$$p_1 - p_2 = 0 \tag{1.38}$$

The impossibility for the inversion can be seen in Figure 2. A entire line on $(\dot{q}_1, \dot{q}_2)$ plane goes into a point in the (p_1, p_2) plane trough the map.

[4] This can be seen in more details in the Appendix A

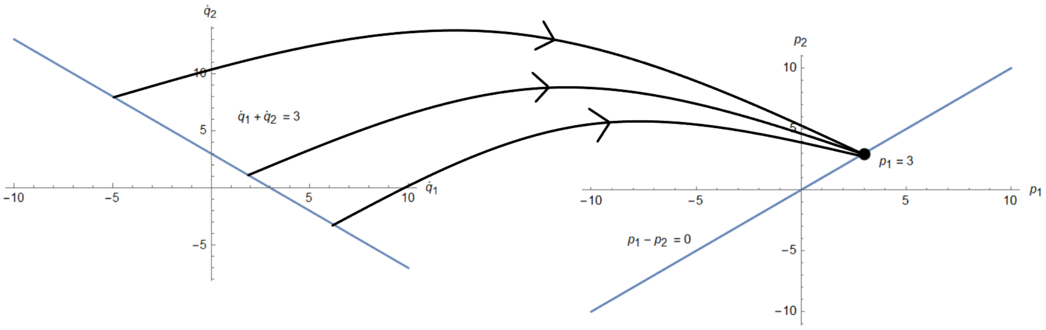

Figure 2. Map from velocity to momentum in a specific case where $p_1 = 3$

We can see already in this system how strange are constrained theories. If we get the Euler-Lagrange equations of motion for this Lagrangian we arrive at:

$$\ddot{q}_1 + \ddot{q}_2 = 0 \tag{1.39}$$

If we fix the boundary conditions to be:

$$q_1(0) = q_2(0) = 0 \tag{1.40}$$

$$\dot{q}_1(0) = \dot{q}_2(0) = \frac{v}{2} \tag{1.41}$$

Then it is clear that one solution for this equation is:

$$q_1(t) = \frac{vt}{2} \tag{1.42}$$

$$q_2(t) = \frac{vt}{2} \tag{1.43}$$

This solution is what we would expect for two free particles as we can see in the Figure 3.

However, there is more freedom than that in this equation of motion. Under the same boundary conditions we could have solutions like shown in the Figure 4

$$q_1(t) = \frac{vt}{2} + \frac{gt^2}{2} \tag{1.44}$$

$$q_2(t) = \frac{vt}{2} - \frac{gt^2}{2} \tag{1.45}$$

In this case the particles appear to accelerate in opposite direction. We could even get crazier things like a oscillatory behavior in Figure 5:

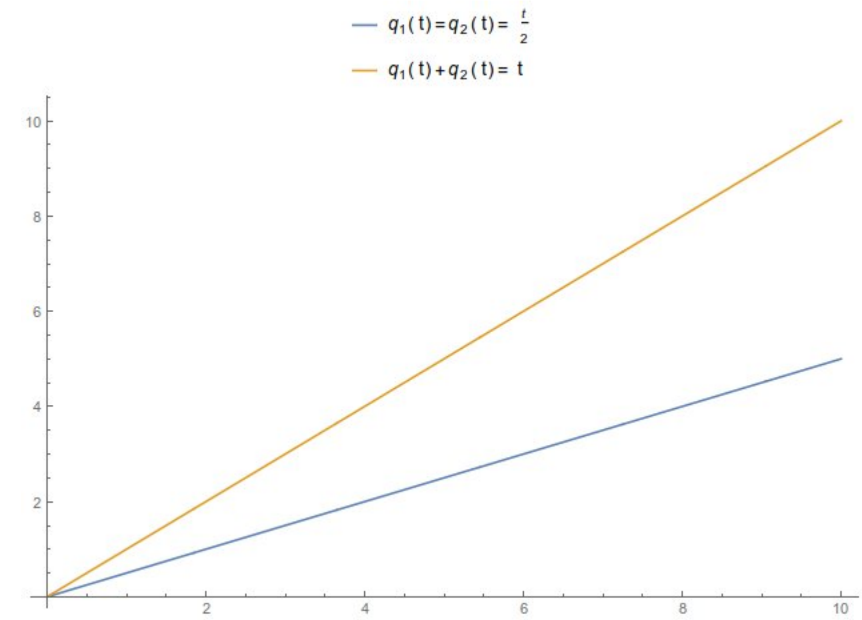

Figure 3. Plot of the solution without any arbitrary function of time.

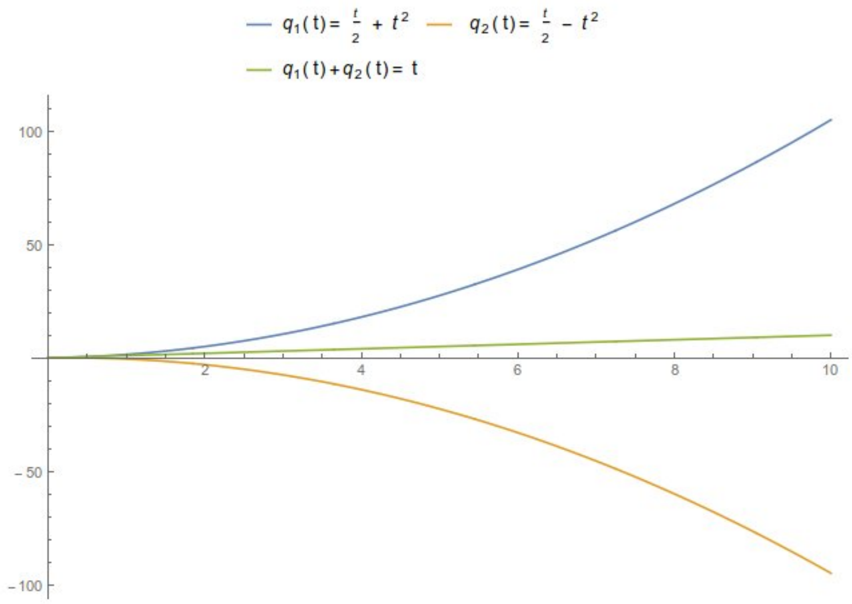

Figure 4. Plot of the solution with a aceleration term.

$$q_1(t) = \frac{vt}{2} + \sin(\omega t) \tag{1.46}$$

$$q_2(t) = \frac{vt}{2} - \sin(\omega t) \tag{1.47}$$

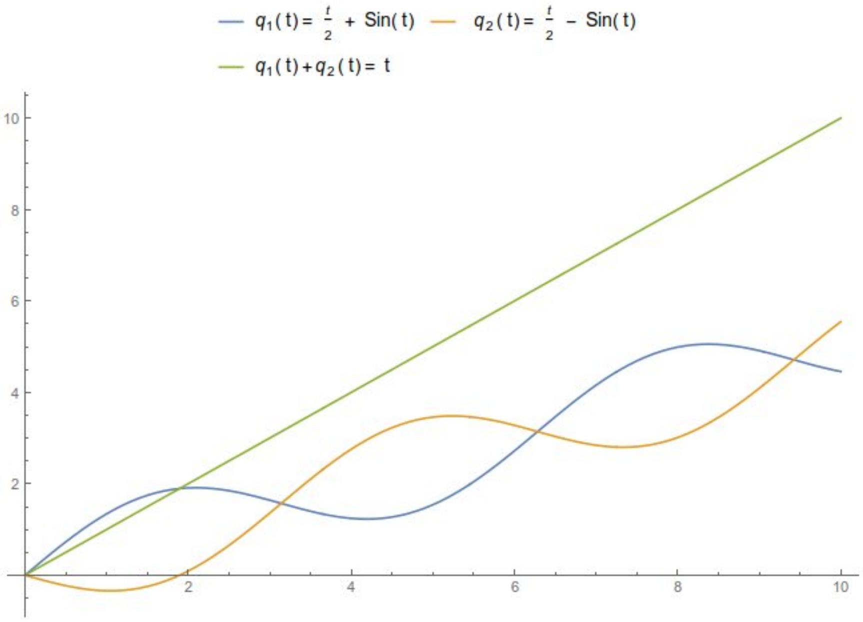

Figure 5. Plot of the solution with a oscilatory term.

In this simple system, we already see the strange nature of a constrained dynamics. We will learn how to deal with such ambiguities ahead but first, we will need to introduce the concept of Lagrange multiplier.

1.3 Lagrange multiplier

The need for adding more objects in the theory can be justified as trying to transform the map into a one to one correspondence. We will insert a new object such that the map became one to one in the new plane defined by (p, λ). Starting in a system with m constraints C_A, the prescription is to add to the Hamiltonian:

$$H_T = H + \lambda(t)^A C_A \tag{1.48}$$

Because our system should evolve only on the constraint surface inside the phase space this addition will not change the equations of motion at this surface. The $\lambda(t)^A$ are the Lagrange multiplier and there are 2 ways to move forward from this. The first one is to enlarge the phase space and include this λ^A as coordinates and solve all the $n+m$ equations. The second one that we will use is to keep the λ^A arbitrary functions of time. In this case, we have n equations of motion but they will depend on arbitrary functions of time. As a consequence, the time evolution will not be unique as we saw in the earlier example. This is not a problem because we can connect all the endpoints of the trajectory, there will be a flow moving through them generated by the constraint. This flow does not change the dynamics and we can identify all the orbits as physically equivalent. This prescription is equivalent in the Lagrangian formalism, as one understands the Lagrangian multiplier as a way to extremize a function subject to a set of constraints:

$$L_T = L - \lambda(t)^A C_A \tag{1.49}$$

We will work in Hamiltonian formalism because we want to quantize the theory, in the end, using the canonical formalism. Because we will work in Hamiltonian formalism we will need to deal with Poisson brackets, there is a problem when one computes the Poisson bracket using the constraint first. It is nicer if we introduce a new notation to talk about things off the constraint surface, just in like the way that we use off-shell quantities in the action formalism.

1.4 Weak and Strong equality

It is clear in equation (1.48) that we are working with quantities that when the equation of motion is used the constraint became satisfied. This off-shell nature makes the computation of Poisson bracket a little strange. A naive computation of a Poisson bracket using the constraint first would give the wrong result, as it would vanish with anything. To make life simpler it was proposed to define the weak equality, something that is only equal in the constraint surface:

$$f(q,p) \approx g(q,p) \tag{1.50}$$

This means that the difference between the functions is:

$$f(q,p) - g(q,p) = \theta^A C_A \tag{1.51}$$

We transform that into strong equality when the constraints are used in this case the functions became equal. So we will use weak equality just to keep all the constraint dependence in the functions of the phase space. Then we actually write the total Hamiltonian as:

$$H_T \approx H + \lambda(t)^A C_A \tag{1.52}$$

Keeping in mind to only use the constraint at the end of the calculation. Using this notation, we can start to analyze the types of constraints.

1.5 Classification of Constrains

When you have a physical system with a set of constraints C_A, we call them primary constraints. Now this set of constraints need to be preserved by time evolution, as discussed before the interpretation of constraint as a conserved quantity. This is not always the case:

$$\{C_A, H_T\} \approx f_A^\theta G_\theta \tag{1.53}$$

$$G_\theta \not\approx 0 \tag{1.54}$$

To make the constraint consistent with time evolution we need to pick this new set of functions and use as constraints so the equation (1.53) became zero trivially. This set of new constraints are called secondary constraints. This consistency check needs to go all the way until you exhaust them. Any new constraint generated by the imposition of the secondary constraint will be called secondary as well. Using this process there are 3 different results that could happen. First is that you arrive in:

$$0 = 0 \tag{1.55}$$

And nothing needs to be done. The second one is when you get a new function of (q^i, p_i) as expressed by G_θ, independent of the others constraints. The procedure as cited above is to use this expression as a new set of constraints. The last possibility is that you impose a condition in the Lagrange multiplier λ. This means that not all the Lagrange multiplier are free. When you make all the consistency check you should end up with a set of primary and secondary constraints:

$$C_A \ , \ A = 1, ..., n + m = k \tag{1.56}$$

This set of k constraints can be treated together as the distinction of primary and secondary is arbitrary. An important difference in the constraints will arise if their Lagrange multiplier is determined or not. To understand this a little deeper, a consistency condition that fixes λ would be something like:

$$\{C_A, H\} + \lambda^B \{C_A, C_B\} \approx 0 \tag{1.57}$$

If this equation fixes λ as functions of (q, p) then:

$$\lambda^A = \Phi^A(q, p) \tag{1.58}$$

But this solution is not unique as we can add the homogeneous solution to this:

$$\sigma^B \{C_A, C_B\} = 0 \tag{1.59}$$

Then the most general solution for the Lagrange multiplier is:

$$\lambda^A = \Phi^A + \xi_m \sigma^{mA} \tag{1.60}$$

Here we write the most general linear independent combination of homogeneous solutions. With this λ fixed we can go back to the initial Hamiltonian and use (1.60):

$$H_T \approx H + \Phi^A C_A + \xi_m \sigma^{mA} C_A \tag{1.61}$$

Writing the Hamiltonian as:

$$H_T \approx H' + \sigma^m C_m \tag{1.62}$$

Where:

$$H' = H + \Phi^A C_A \tag{1.63}$$

$$C_m = \Sigma^{mA} C_A \tag{1.64}$$

This has an interesting consequence, we started with arbitrary Lagrange multipliers but turned out that not all of them were arbitrary. After we deal with them there still some arbitrary Lagrange multipliers related to the homogeneous solution. It is possible then to define 2 kinds of constraints. When you have a constraint C, if the Poisson bracket with all others constraints is weakly zero this is called a first class constraint:

$$\{C, C_A\} \approx 0 \tag{1.65}$$

If the constraint is first class we will use indexes m,n,p. Otherwise, the constraint is called second class and we will use indexes α, β, γ. This distinction is important as they play very different physical roles. The number of independent functions of time appearing in the total Hamiltonian is equal to the number of first class constraints. In a more High Energy terminology, one can say that first class constraints are gauge redundancy and the constraint generate a gauge transformation. The next lecture we will investigate deeper this statement and the two types of constraints and how to deal with them classically.

2 Lecture II

2.1 First class constraint

Lets work out a case where we have only first class constraints, this means that under the set of constraints C_m, all of them:

$$\{C_m, C_n\} \approx 0 = f^p_{mn} C_p \tag{2.1}$$

As said before when writing the Hamiltonian we would have something like:

$$H_T \approx H + \lambda(t)^m C_m \tag{2.2}$$

All of the Lagrange multipliers would be indeed arbitrary. This means that the time evolution is not uniquely fixed by the initial conditions:

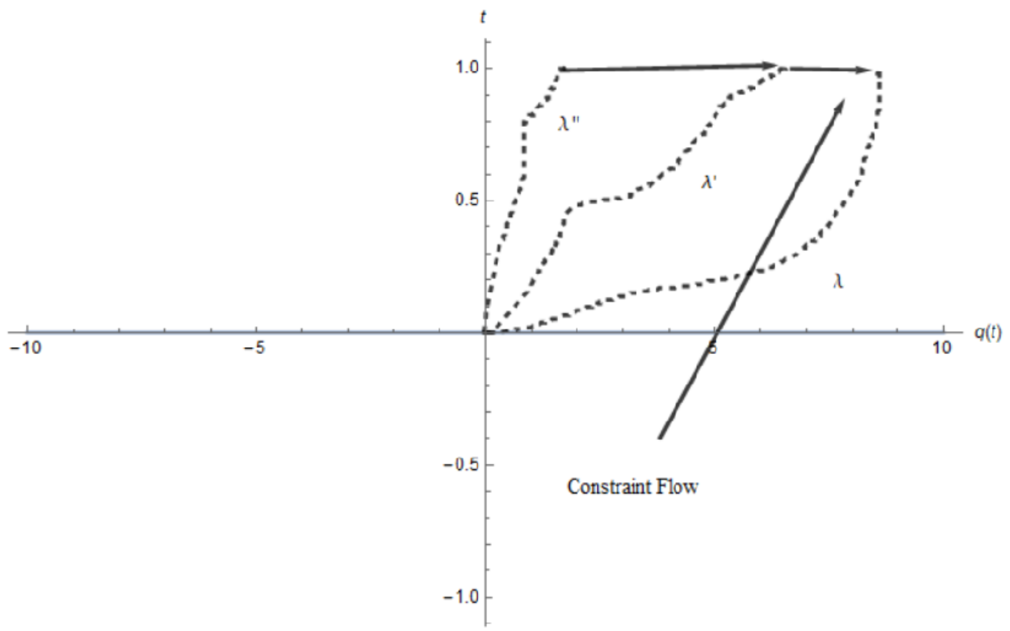

Figure 6. Evolution of a system with first class constraint depending on the choice in $t = 0$ of the arbitrary function.

This is not a problem because a physical observable does not depend on these arbitrary functions. For gauge-invariant (physically measurable quantities) quantities, all of the Hamiltonian should give the same time evolution, since they are all weakly equivalent. It is only for non-gauge-invariant quantities that the distinction becomes important. Having arbitrary functions of time $\lambda^m(t)$ in the total Hamiltonian gives us the hint that not all phase space coordinates are observable. This means that a physical state can be determined uniquely once a set of phase space coordinates is given but one can't reconstruct the physical state from the coordinate information alone. There is more than one set of variables representing a chosen physical state. In classical mechanics, it is expected that given an initial set of phase space coordinates at a time t_0 we can use the equations of motion to determine the physical state at other times. This deterministic nature makes us understand that any ambiguity in this process can't be physically relevant. In an instant of time t the variables will depend on the choice of the arbitrary functions λ. We can look for the difference of the evolution from a phase space function with a selected λ at the time t_0, F_λ and a different function λ', $F_{\lambda'}$. The time evolution of F is in general of the form:

$$\dot{F} = \{F, H_0\} + \lambda^m \{F, C_m\} \tag{2.3}$$

In a infinitesimal time interval we have:

$$F_\lambda(t) = F(t_0) + \frac{dF}{dt}|_{t=t_0}(t - t_0) + ... \tag{2.4}$$

Using the time evolution one has:

$$\frac{dF_\lambda}{dt}|_{t=t_0} = \{F(t_0), H_0\} + \lambda^m(t_0)\{F(t_0), C_m\} \tag{2.5}$$

Then the difference between the phase space function at the time t from different functions is:

$$F_\lambda - F_{\lambda'} = (t - t_0)(\lambda^m(t_0) - \lambda'^m(t_0))\{F(t_0), C_m\} \tag{2.6}$$

We can write this as

$$\delta_\epsilon F = \epsilon^m(t)\{F, C_m\} \tag{2.7}$$

Where we identified:

$$\epsilon^m(t) = (t - t_0)(\lambda^m(t_0) - \lambda'^m(t_0)) \tag{2.8}$$

Because of the arguments above this transformation does not change physical sate at the time t. Such a transformation is called gauge transformation. This gives us that first class constraint generate gauge transformation. We can see if this transformations form a algebra:

$$[\delta_\epsilon, \delta_\mu]F = \{\{\epsilon^m C_m, \mu^n C_n\}, F\} \tag{2.9}$$

Because these are first class constraints one has:

$$\{\epsilon^m C_m, \mu^n C_n\} = \epsilon^m \mu^n f^p_{mn} C_p \tag{2.10}$$

So we can see that the algebra closes because:

$$\theta^p = \epsilon^m \mu^n f^p_{mn} \tag{2.11}$$

$$[\delta_\epsilon, \delta_\mu]F = \delta_\theta F \tag{2.12}$$

This is true even if the structure constants depend on the Phase Space variables. In general, is not possible to prove that every first class constraint generates gauge transformation(Dirac Conjecture states that every secondary first class constraint generates gauge transformation.)[5]. Nevertheless for what follows one can assume that. For all physical applications so far this is true and it is not know how to quantize a theory if it is not the case. Just a final reminder these considerations were made by identifying the time t as an observable. If one work with reparametrization invariant theory we have to be more careful as we will see in the Lecture 4.

The constraint algebra generates a flow linking all of the equivalent solutions and in this point, we could do two things. The first is choose a representative in each orbit and do calculations remembering that the physics is invariant under the choice of representative. This process is in the heart of gauge fixing as we will see in a moment. The second one is to treat your system without gauge fixing and find gauge invariant objects that describe the system:

$$\{A(q,p), C_m\} \approx 0 \tag{2.13}$$

This objects will be invariant under gauge transformations and can be used as observable. The first one is usually used in classical physics and the second one not so much. Going to quantum mechanics the first one became a little trickier and sometimes the second is easier, as we will discuss when we quantize the system. Now let's analyze the second class constraint case and after discuss the role of the action in this formalism.

2.2 Second class constraint

We already saw that if a constraint has a weakly non vanishing bracket with the others constraints they are second class:

$$\{C^2_\alpha, C^2_\beta\} \neq 0 \tag{2.14}$$

It is possible that inside a set of constraints C_A such that:

$$\det(\{C_A, C_A\}) \approx 0 \tag{2.15}$$

That in a first glance is first class but inside them could exist a sub-set of second class constraints:

$$\{C^2_\alpha, C^2_\beta\} \neq 0 \tag{2.16}$$

Such that:

$$\det\big(\{C^2_\alpha, C^2_\beta\}\big) \neq 0 \tag{2.17}$$

[5]In fact it is know examples of systems where this is not the case like: $L = \frac{1}{2}e^y\dot{x}^2$.

It is always possible to separate them by changing the basis such that the matrix reads weakly as:

$$\{C_A, C_B\} \approx \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{\alpha\beta} \end{bmatrix} \tag{2.18}$$

Where the first entry are the first class constraint C^1_m and the second entry are the second class C^2_α. It should be noted that you could lose the manifestation of some symmetry in this process. This separation is not unique as it is preserved by the redefinition's:

$$C^1_m \to \Theta^n_m C^1_n \tag{2.19}$$

$$C^2_\alpha \to \Theta^\beta_\alpha C^2_\beta + \Theta^m_\alpha C^1_m \tag{2.20}$$

Given that all Θ's are invertible matrices. We will assume that the matrix $\Lambda_{\alpha\beta}$ is invertible everywhere in the surface of the second class constraint $C^2_\alpha = 0$, not needing the full set of constraints. This will be necessary to deal with second class constraint. Different from first class constraint, these constraints cannot be interpreted as gauge generators. In fact, they are not generators of any physical transformation. If this is not the case, how we treat second class constraints? To answers this question we can look for a simplified system that has this propriety. Consider a system with a set of constraints:

$$C^2_1 = q^1 = 0 \tag{2.21}$$

$$C^2_2 = p_1 = 0 \tag{2.22}$$

They are indeed second class:

$$\{C^2_1, C^2_2\} = 1 \tag{2.23}$$

It is straightforward what to do in this case. The constraints tell us that (q^1, p_1) are not important and we can just discard them. The Poisson bracket get modified such that:

$$\{F, G\}_{DB} = \sum_2^n \left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i} \right) \tag{2.24}$$

This new bracket respects the constraint and its evolution. Most important the bracket between a constraint and any phase space function is strongly zero. This gives us a hint that if we work using these brackets one can use only strong equality and set the constraints to zero. The equation of motion of the remaining variables are the same using this new bracket and it has all the nice proprieties that one expect of a Poisson bracket. Using this expertise it is possible to generalize this concept for an arbitrary set of second class constraints, as was done by Dirac. He realized that given a subspace of second class constraint that has an invertible matrix:

$$\Lambda^{-1}.\Lambda = \mathbb{I} \tag{2.25}$$

We can define a modified bracket:

$$\{F,G\}_{DB} = \{F,G\} - \{F,C^2_\alpha\}(\Lambda^{-1})^{\alpha\beta}\{C^2_\beta,G\} \tag{2.26}$$

This is called Dirac Bracket and has all the important proprieties that a Poisson bracket needs to have while enforcing the second class constraints:

- Antisymmetry: $\{F,G\}_{DB} = -\{G,F\}_{DB}$.
- Linearity(In both slots): For z_1,z_2 constants $\in \mathbb{C}$, $\{z_1F_1+z_2F_2,G\}_{DB} = z_1\{F_1,G\}_{DB}+ z_2\{F_2,G\}_{DB}$.
- Null element: For z_1 constant, $\{z_1,F_1\}_{DB} = 0$.
- Jacobi identity: $\{A,\{B,C\}_{DB}\}_{DB} + \{C,\{A,B\}_{DB}\}_{DB} + \{B,\{C,A\}_{DB}\}_{DB} = 0$.
- Leibniz's rule(In both slots): $\{A,BC\}_{DB} = \{A,B\}_{DB}C + B\{A,C\}_{DB}$
- Enforces all the second class constraints: $\{C^2_\alpha,F\}_{DB} = 0$ for any $F(q,p)$
- For a first class function its Poisson: $\{C^1_m,F\}_{DB} \approx \{C^1_m,F\}$
- For F and G first class functions: $\{R,\{F,G\}_{DB}\}_{DB} \approx \{R,\{F,G\}\}$

In a geometrical formulation, this bracket is the symplectic form on the constrained surface and dictates the dynamics pairing with the Hamiltonian. If one can construct such a bracket then the treatment of the constraint is easy, just set them to be strongly zero and work with the Dirac Brackets. Because the extended Hamiltonian is first class it still gives the right equation of motion:

$$\dot{F} \approx \{F,H_E\} \approx \{F,H_E\}_{DB} \tag{2.27}$$

It is possible to work with only Dirac brackets even if there is a first class constraint subset. The main problem is that even that all the constraint information is inside the Dirac bracket it is not always easy to find its explicit form. Even harder when going to quantum mechanics as one needs to find a realization of such a bracket. Because of that, it is normal to want to deal with second class constraints in different ways that bypass this formulation. The only known method is to add an additional degree of freedom in a way that you see the system as a first class constraint and recover the system by a gauge fixing. Before doing that let's just see the role of the Lagrange multipliers in the case of second class constraint. Consider now a system with a second class constraint, in such a system the Lagrange multipliers are all determined by the relation used in the example above:

$$\lambda_\alpha \approx -\Lambda^{\alpha\beta}\{C_\beta,H\} \tag{2.28}$$

This means that we can substitute back this solution on the first Hamiltonian and our coordinates will give the right physical description:

$$H_T \approx H - C_\alpha\Lambda^{\alpha\beta}\{C_\beta,H\} \tag{2.29}$$

This expression can be a mess and it is recommended to go through the Dirac Bracket route as one can use the constraints from the start.

2.3 Gauge Fixing

Studying first class constraint and the gauge freedom associated with it made clear that there is more than one set of canonical variables that correspond to a chosen physical state. This ambiguity can be used to our advantage to describe the system in a clever way. Sometimes this ambiguity can be a pain to deal with, having too much more variables than needed obscuring the physical system. Thinking in this manner one can get rid of this ambiguity in a consistent manner if we impose further restrictions on the system in a way that there is a one-to-one correspondence between a state and the canonical variables. We already know that the physics is invariant under the choice of representative, in other words, the choice of λ^m. The gauge condition is just a way to avoid overcount of a state, because of this nature, it is a thing coming from outside the theory. We are allowed to do that as we only remove the non-observable arbitrary elements of the theory and don't affect the gauge invariant objects. In a first glance the gauge condition could be any function of the phase space, one for each first class constraint:

$$G_m(q,p) \approx 0 \tag{2.30}$$

However, a good gauge condition needs to satisfy some conditions:

- The gauge must be accessible: Given a set of canonical variables, there must exist a gauge transformation that maps then into one that satisfies the gauge fixing. This means that given a gauge slice:

  $$G_m(q,p) = 0 \tag{2.31}$$

  If a set of coordinates (q', p') does not respect this condition we should be able to find a set (q,p) that does it. Then we should be able to find a gauge transformation that takes the coordinates to the gauge surface

  $$G_m(\delta_\epsilon q', \delta_\epsilon p') = 0 \tag{2.32}$$

- The condition must fix the gauge completely. This means that:

  $$\lambda^m \{G_n, C_m\} \approx 0 \tag{2.33}$$

  Must imply:

  $$\lambda^m \approx 0 \tag{2.34}$$

These conditions guarantee that the set of first class constraints with gauge conditions are now second class and the gauge fixing don't change the physics. In the end, there is no first class constraint left and one can treat the system as a second class one with the associated Dirac Brackets. We can have a geometrical interpretation, the gauge fixing surface:

$$G_m(q,p) = 0 \tag{2.35}$$

Should intersect the gauge orbits, which lie in the constraint surface only once. The following condition guarantees this locally:

$$\det(\{G_m, C_n\}) \neq 0 \tag{2.36}$$

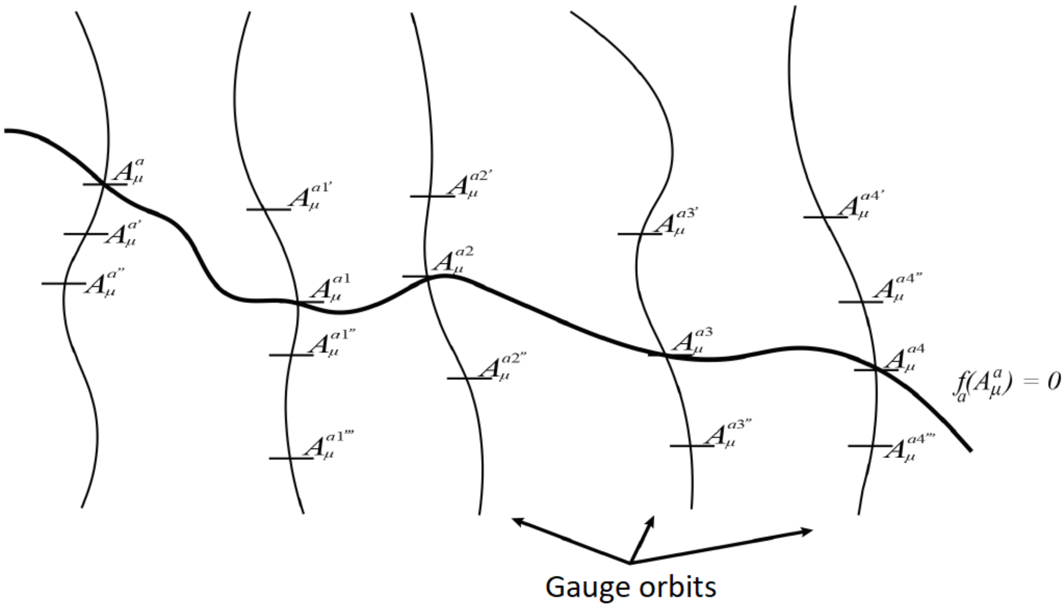

Figure 7. Representation of gauge orbits and a good gauge fixing.

This does not guarantee that this is respect globally. The geometry of the constraint surface could forbid the existence of a global gauge condition. This is known as the Gribov problem, and for example, affects linear covariant gauges in non-abelian Yang-Mills theories. This is one reason that developing a theory of first class constraint without gauge fixing is interesting.

This description of gauge fixing a first class constraint system and transforming into a second class gives us the possibility to treat second class constraints differently. One could try to find an enlarged system that when gauge fixed gives us the second class constraint one. Then we treat directly the first class constraint without gauge fixing and avoiding the Dirac Bracket. This process can always be done but the removal of second class constraint is not unique and could spoil some manifest symmetry. Just to see an example, the second class constraint system talked before:

$$q^1 \approx 0 \tag{2.37}$$

$$p_1 \approx 0 \tag{2.38}$$

Can be seen as the first class constraint system:

$$p_1 \approx 0 \tag{2.39}$$

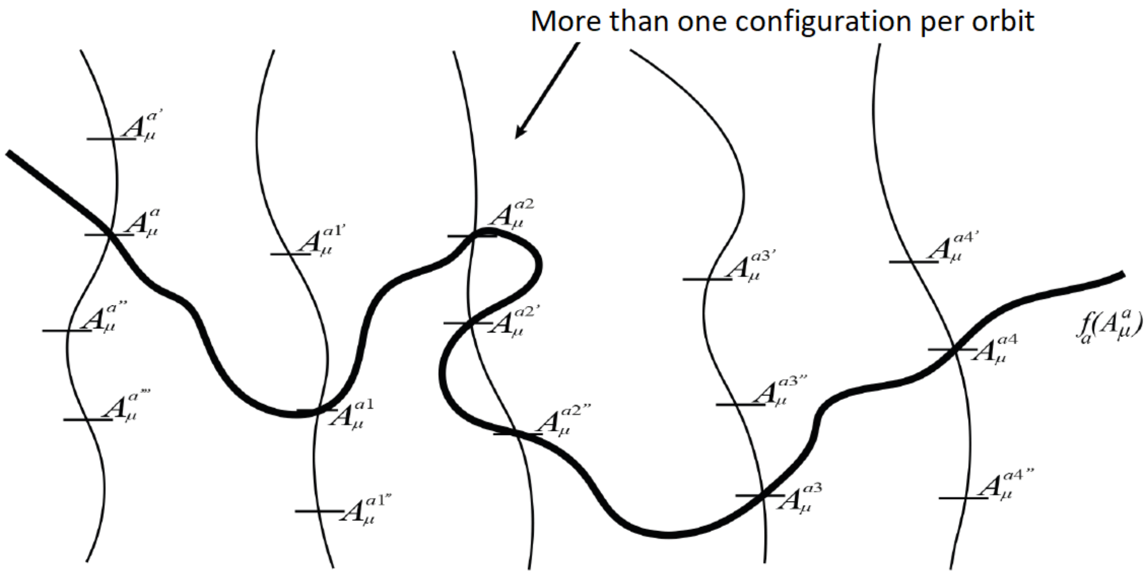

Figure 8. Representation of gauge orbits and a bad gauge fixing.

With the gauge transformation:

$$q^1 \rightarrow q^1 + \lambda \tag{2.40}$$

And gauge fixing:

$$q^1 \approx 0 \tag{2.41}$$

The advantage of this method is not having to deal with Dirac Bracket. When going to quantum mechanics this can be a huge advantage.

Before we finish and work some classical examples lets see how the count of degree of freedom goes in a theory with constraint. Since after the gauge fixing there are no more redundancy lefts we have:

$$2 \times (\text{Number of Physical degrees of freedom}) = (\text{Number of independent canonical variables}) = \tag{2.42}$$

$$= (\text{Total Number of canonical variables}) - (\text{Number of second class constraints}) -$$

$$-(\text{Number of first class constraints}) - (\text{Number of gauge fixing})$$

We can write in the compact form:

$$2 \times (\text{Number of Physical degrees of freedom}) = (\text{Total Number of canonical variables}) - \tag{2.43}$$

$$-(\text{Number of second class constraints}) - 2(\text{Number of first class constraints}) \tag{2.44}$$

This relation is well defined and always holds for systems with finite degrees of freedom. Turns out that can exist complications on the case of infinite degrees of freedom. Going to the continuum we need to be careful as to how arbitrary the Lagrange multipliers can be in the first class case, they appear in the Hamiltonian as:

$$\int \mathrm{d}^d x \, \lambda^m(x,t) C^1_m(x) \tag{2.45}$$

The only assumption is that the transformation generated by the constraint is a gauge transformation:

$$\delta F = \int \mathrm{d}^d x \, \lambda^m(x,t) \{F, C^1_m(x)\} \tag{2.46}$$

However, this has no general solution and we need to work case by case. Depending on the nature of λ it is not guaranteed that this transformation does not change the physical states. Normally the most general λ need to be such that:

$$\int \mathrm{d}^d x \, \lambda^m(x,t) C^1_m(x) \approx 0 \tag{2.47}$$

Implies

$$C^1_m(x) \approx 0 \tag{2.48}$$

For instance, a gauge field that vanishes at infinite, if one uses a λ that is constant at infinity it will give rise to an overall charge rotation. Invariance under these transformations selects zero total charge states. This tells us that the total charge became a new constraint of the system which generates its own new gauge invariance. They may, for instance, exist transformations that are not continuously connected to the identity (Large gauge transformations). They again map goods states into good states but seeing them as proper gauge transformations amount to an additional assumption. This will not be a problem in the cases worked here but is good to be known. The last thing that we will work out in this lecture, before the examples, is how to relate these first class constraints to the gauge symmetry of the Lagrangian formalism, more specifically, the action.

2.4 Gauge invariance of the action

In the start of the treatment of constraint we stated with the action on first order formalism:

$$S_L = \int \mathrm{d}t \, L = \int \mathrm{d}t \left(\dot{q}^i p_i - H\right) \tag{2.49}$$

The impossibility to construct the Hamiltonian made us use the Lagrange multiplier, the action then became:

$$S_L \approx \int \mathrm{d}t \left(\dot{q}^i p_i - H_0 - \lambda^A C_A\right) \tag{2.50}$$

This action has the same gauge symmetries from the Lagrangian formalism. However, doing the consistency check for the constraint we saw that there is more constraint than initially enforced, this made us add those secondary constraints in the same way that we did the first one. In this step the equations of motion changes, but we can't distinguish this because of the redundancy of the equations. This process of add the secondary constraints create new gauge symmetries, this can be seen as we can always change the additional Lagrange multipliers. As it is expected more arbitrary multiplier implies more gauge symmetry. This is what we will call extended action:

$$S_E \approx \int \mathrm{d}t \left(\dot{q}^i p_i - H_0 - \lambda^A C_A - \lambda^{\theta}_{secondary} C^{secondary}_{\theta}\right) \tag{2.51}$$

Lets work out a little more the most general case. We will see that the Lagrange multipliers need to transform in order to render the action invariant and sometimes we can use theses transformations to find the algebra between the constraints. For now we will work with the primary and secondary constraint together only differentiating between first class and second one:

$$S_E \approx \int \mathrm{d}t \left(\dot{q}^i p_i - H_0 - \lambda_1^m C^1_m - \lambda_2^{\alpha} C^2_{\alpha}\right) \tag{2.52}$$

The gauge transformations are:

$$\delta_\epsilon \equiv \epsilon^m(t)\{\_, C^1_m\} \tag{2.53}$$

$$\delta_\epsilon q^i \approx \epsilon^m(t)\{q^i, C^1_m\} = \epsilon^i(t)\frac{\partial C^1_m}{\partial p_i} \tag{2.54}$$

$$\delta_\epsilon p_i \approx \epsilon^m(t)\{p_i, C^1_m\} = -\epsilon^m(t)\frac{\partial C^1_m}{\partial q^i} \tag{2.55}$$

The most general algebra that can happen is:

$$\{C^1_m, C^1_n\} \approx f^p_{mn} C^1_p + g^{\alpha\beta}_{mn} C^2_{\alpha} C^2_{\beta} \tag{2.56}$$

$$\{C^1_m, C^2_{\alpha}\} \approx h^n_{m\alpha} C^1_n + u^{\beta}_{m\alpha} C^2_{\beta} \tag{2.57}$$

$$\{C^2_{\alpha}, C^2_{\beta}\} \approx K_{\alpha\beta} \tag{2.58}$$

$$\{C^1_m, H_0\} \approx d^n_m C^1_n + e^{\alpha\beta}_m C^2_{\alpha} C^2_{\beta} \tag{2.59}$$

$$\{C^2_{\alpha}, H_0\} \approx r^m_{\alpha} C^1_m + t^{\beta}_{\alpha} C^2_{\beta} \tag{2.60}$$

With $e_m^{\alpha\beta}$ and $g_{mn}^{\alpha\beta}$ symmetric in the α,β. The second class constraint appears only quadratic in the first and third relation because the bracket of two first class constraint is first class as well. And of course:

$$\det K_{\alpha\beta} \neq 0 \tag{2.61}$$

We could get rid of the second class constraint using the Dirac bracket, this would change the relations, simplifying its form:

$$\{C_m^1, C_n^1\}_{DB} = \tilde{f}_{mn}^p C_p^1 + \tilde{g}_{mn}^{\alpha\beta} C_\alpha^2 C_\beta^2 \tag{2.62}$$

$$\{C_m^1, C_\alpha^2\}_{DB} = 0 \tag{2.63}$$

$$\{C_\alpha^2, C_\beta^2\}_{DB} = 0 \tag{2.64}$$

$$\{C_m^1, H_0\}_{DB} = \tilde{d}_m^n C_n^1 + \tilde{e}_m^{\alpha\beta} C_\alpha^2 C_\beta^2 \tag{2.65}$$

$$\{C_\alpha^2, H_0\}_{DB} = 0 \tag{2.66}$$

Now we can analyze what are the transformations of the Lagrange multipliers such that the extended action is invariant:

$$\delta_\epsilon S_E = 0 = \int \mathrm{d}t \left(\delta_\epsilon(\dot{q}^i p_i) - \delta_\epsilon H_0 - (\delta_\epsilon \lambda_1^m) C_m^1 - \lambda_1^m(\delta_\epsilon C_m^1) - (\delta_\epsilon \lambda_2^\alpha) C_\alpha^2 - \lambda_2^\alpha(\delta_\epsilon C_\alpha^2)\right) \tag{2.67}$$

The first one give rise to a boundary term plus a extra:

$$\delta_\epsilon(\dot{q}^i p_i) = \frac{d}{dt}(\delta_\epsilon q^i) p_i + \dot{q}^i \delta_\epsilon p_i = \frac{d}{dt}(\delta_\epsilon q^i p_i) - \delta_\epsilon q^i \dot{p}_i + \dot{q}^i \delta_\epsilon p_i \tag{2.68}$$

Using the transformation of the canonical variables and remembering that ϵ has time dependence we can write:

$$\delta_\epsilon(\dot{q}^i p_i) = \frac{d}{dt}\left(\epsilon(t)^m(\frac{\partial C_m^1}{\partial p_i} p^i - C_m^1)\right) + \dot{\epsilon}^m C_m^1 \tag{2.69}$$

The rest of the transformations are straightforward and one can conclude the transformations of the Legendre multipliers as:

$$\delta_\epsilon \lambda_1^m = \dot{\epsilon}^m + \lambda_1^n \epsilon^l f_{nl}^m + \lambda_2^\alpha \epsilon^l h_{\alpha l}^m - \epsilon^n d_n^m \tag{2.70}$$

$$\delta_\epsilon \lambda_2^\alpha = \lambda_1^m \epsilon^n g_{mn}^{\alpha\beta} C_\beta^2 + \lambda_2^\beta \epsilon^m u_{m\beta}^\alpha - \epsilon^m e_m^{\alpha\beta} C_\beta^2 \tag{2.71}$$

Where the boundary term is said to be zero by appropriate boundary conditions of the gauge transformation. This can be used to find the coefficients if one knows the transformation of the Lagrange multipliers, that can be guessed by geometrical means.

If we want to recover the gauge symmetries from the initial Lagrangian we just need to gauge fix these additional symmetries:

$$\lambda^{\theta}_{secondary} = 0 \tag{2.72}$$

In such a way that:

$$\delta\lambda^{\theta}_{secondary} = 0 \tag{2.73}$$

In this sense, the gauge symmetry in the Lagrangian formalism is the residual gauge symmetry of the extended action from the Hamiltonian formalism. We need to be careful when comparing the Lagrangian and Hamiltonian formalism. In the end, both formalisms give the same observables and answers but in the middle things are different.

2.5 Example First Class Constraint: Gauge redundancy

The case that we will work out is the following:

$$\mathcal{L} = \frac{m}{2}(\dot{q}_1 + \dot{q}_2)^2 - V(q_1 + q_2) \tag{2.74}$$

This system is good to illustrate what sometimes is possible to do with constrained systems. We already saw in the last lecture that the solutions for the equation of motion are not unique in this case. It is easy to see that if we go to the center of mass coordinate this would be a simple particle moving in a potential. This coordinate is what is called unconstrained variable. We could solve the constraint in some sense only by an appropriate choice of coordinates. If one finds such coordinate then they can proceed normally describing the system. For the sake of understanding a little better first class constraint, let's ignore this change and work out the Hamiltonian formulation. The canonical momentum in this case are:

$$p_1 = m(\dot{q}_1 + \dot{q}_2) \tag{2.75}$$

$$p_2 = m(\dot{q}_1 + \dot{q}_2) \tag{2.76}$$

Its straightforward to see that we have a constraint:

$$C = p_2 - p_1 \tag{2.77}$$

And the Hamiltonian can be written as:

$$H \approx p_1\dot{q}_1 + p_2\dot{q}_2 - \mathcal{L} \tag{2.78}$$

$$H \approx \frac{p_1^2}{2m} + \dot{q}_2(p_2 - p_1) + V(q_1 + q_2) \tag{2.79}$$

We can see that $\dot{q}_2$ plays the role of the Lagrange multiplier. The Hamiltonian is written then as:

$$H_T \approx \frac{p_1^2}{2m} + V(q_1 + q_2) + \lambda(p_2 - p_1) \tag{2.80}$$

It is clear that this constraint is first class and does not generate any other constraint when we go trough consistency condition:

$$\{p_2 - p_1, p_2 - p_1\} = 0 \tag{2.81}$$

And for the time evolution:

$$\{\frac{p_1^2}{2m} + V(q_1 + q_2) + \lambda(p_2 - p_1), p_2 - p_1\} = 0 \tag{2.82}$$

Because V does not depend on the difference of position. It is clear in this example the nature of first class constraint, they are redundancy in our description. There is a particle in one dimension whose position is $(q_1 + q_2)$, but it has some internal degree of freedom which can be seen as $(q_2 - q_1)$. The time evolution of this degree of freedom is arbitrary. Logically to have a deterministic classical theory we have to assume that this degree of freedom is nonphysical, and consequentially cannot be measured. Two states, which can evolve out of a single state in this way must be declared to be physically equivalent, and a transformation that link these states is called a gauge transformation. It is important to note that this redundancy does not come out because we didn't impose enough initial conditions. Lets see the gauge orbits of this case and some gauge invariant objects. The time evolution can be computed from the total Hamiltonian:

$$\dot{q}^1 \approx \{q^1, H_T\} = \frac{p_1}{m} - \lambda \tag{2.83}$$

$$\dot{q}^2 \approx \{q^2, H_T\} = \lambda \tag{2.84}$$

$$\dot{p}_1 \approx \{p_1, H_T\} = -\frac{\partial V(q_1 + q_2)}{\partial q^1} \tag{2.85}$$

$$\dot{p}_2 \approx \{p_2, H_T\} = -\frac{\partial V(q_1 + q_2)}{\partial q^2} = -\frac{\partial V(q_1 + q_2)}{\partial q^1} \tag{2.86}$$

We can see that the evolution of (q^1, q^2) has the arbitrary function of time, while the momentum don't. The momentum are gauge invariant in this case as one can verify. Because of the nature of the constraint it is easy to construct another gauge invariant object:

$$O_1 = q^1 + q^2 \tag{2.87}$$

$$O_2 = p_1 + p_2 \tag{2.88}$$

In such a way that:

$$\dot{O}_1 = \frac{p_1}{m} = \frac{p_1 + p_2}{2m} = \frac{O_2}{2m} \tag{2.89}$$

$$\dot{O}_2 = -2\frac{\partial V}{\partial q^1} = \frac{\partial V(O_1)}{\partial O_1} \tag{2.90}$$

The gauge transformations are:

$$\delta q^1 = -\lambda \tag{2.91}$$

$$\delta q^2 = \lambda \tag{2.92}$$

$$\delta p_1 = 0 \tag{2.93}$$

$$\delta p_2 = 0 \tag{2.94}$$

Let's see a example of gauge fixing in this case, we need a function of phase space such that:

$$\{p_2 - p_1, f\} \approx G \neq 0 \tag{2.95}$$

We could choose for instance:

$$f = q_2 - A \ , \ \ A \in \mathbb{R} \tag{2.96}$$

This gauge fixing commute with the potential and does not generate any new constraint. Lets work this gauges:

$$H_T \approx \frac{p_1^2}{2m} + V(q_1 + q_2) + \lambda^1(p_2 - p_1) + \lambda^2(q_2 - A) \tag{2.97}$$

Now under time evolution we don't get any new constrain and we get the commutation between them as:

$$\{(p_2 - p_1), (p_2 - p_1)\} = 0 \tag{2.98}$$

$$\{(q_2 - A), (q_2 - A)\} = 0 \tag{2.99}$$

$$\{(p_2 - p_1), (q_2 - A)\} = -1 \tag{2.100}$$

In matrix form we have:

$$\{C_\alpha, C_\beta\} = \mathcal{K}^{-1}_{\alpha\beta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.101}$$

The inverse is:

$$\mathcal{K}_{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.102}$$

Now the Lagrange multipliers are determinate:

$$\lambda^{\alpha} \approx -\mathcal{K}^{\alpha\beta}\{C_{\beta}, H\} \tag{2.103}$$

This means that:

$$\lambda_1 \approx -\{C_2, H\} \tag{2.104}$$

$$\lambda_2 \approx \{C_1, H\} \tag{2.105}$$

Where:

$$H = \frac{p_1^2}{2m} + V(q_1 + q_2) \tag{2.106}$$

And:

$$\{C_1, H\} = 0 \tag{2.107}$$

$$\{C_2, H\} = 0 \tag{2.108}$$

So both Lagrange multipliers are uniquely determined and are zero. The total Hamiltonian can be written as:

$$H = \frac{p_1^2}{2m} + V(q_1 + A) \tag{2.109}$$

Now all constraints are used and this is what one naively would get by an appropriate change of variables.

2.6 Example Second Class Constraint: Particle in a circle

Lets work the system moving on a surface of a circle:

$$\mathcal{L} = \frac{m}{2}(\dot{q_1}^2 + \dot{q_2}^2) \tag{2.110}$$

Going to the Hamiltonian formalism we could add the Lagrange multiplier from the start and enlarge the phase space or use the technique described before. In this case lets go to the Hamiltonian description without any barrier and force the constraints there:

$$H_T \approx \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \lambda(q_1^2 + q_2^2 - r^2) \tag{2.111}$$

Now lets see if the constraint is preserved trough time evolution:

$$\{(q_1^2 + q_2^2 - r^2), \frac{p_1^2}{2m} + \frac{p_2^2}{2m}\} = \frac{(q_1 p_1 + q_2 p_2)}{m} \tag{2.112}$$

This is not zero so by the consistency condition we need to include this as a new constraint:

$$H_T \approx \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \lambda^1(q_1^2 + q_2^2 - r^2) + \lambda^2(q_1 p_1 + q_2 p_2) \tag{2.113}$$

Now the first constraint is preserved trough time evolution, lets see what the second one tells us:

$$\{(q_1p_1+q_2p_2), \frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\lambda^1(q_1^2+q_2^2-r^2)\} = \frac{p_1^2}{m}+\frac{p_2^2}{m}-2\lambda^1(q_1^2+q_2^2) \tag{2.114}$$

This last one fixes λ_1 to be consistent such that:

$$\lambda^1 \approx \frac{1}{2m}\frac{p_1^2+p_2^2}{q_1^2+q_2^2} \tag{2.115}$$

The evolution of the first constraint now that C_2 was added with the new total Hamiltonian will give:

$$\{(q_1^2+q_2^2-r^2), \frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\lambda^2(q_1p_1+q_2p_2)\} = \frac{(q_1p_1+q_2p_2)}{m}+2\lambda_2(q_1^2+q_2^2) \tag{2.116}$$

This fixes the last Lagrange multiplier:

$$\lambda^2 \approx \frac{q_1p_1+q_2p_2}{2m(q_1^2+q_2^2)} \tag{2.117}$$

Our dynamic is now fixed and we can plug this back on the Hamiltonian:

$$H_T = \frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{1}{2m}\frac{p_1^2+p_2^2}{q_1^2+q_2^2}(q_1^2+q_2^2-r^2)+\frac{q_1p_1+q_2p_2}{m(q_1^2+q_2^2)}(q_1p_1+q_2p_2) \tag{2.118}$$

And write the Hamiltonian as:

$$H_T \approx \frac{p_1^2}{2m}(1+\frac{q_1^2+q_2^2-r^2}{q_1^2+q_2^2}+\frac{2q_1^2}{q_1^2+q_2^2})+\frac{p_2^2}{2m}(1+\frac{q_1^2+q_2^2-r^2}{q_1^2+q_2^2}+\frac{2q_2^2}{q_1^2+q_2^2})+\frac{2q_1q_2p_1p_2}{m(q_1^2+q_2^2)} \tag{2.119}$$

And a prior find the equations of motion. Of course this is ugly and for instance the usage of Dirac bracket simplifies a bit, because in this framework we only use strong equality, the Hamiltonian would only be:

$$H = \frac{p_1^2}{2m}+\frac{p_2^2}{2m} \tag{2.120}$$

But the bracket would be modified to the Dirac ones:

$$\mathcal{K}_{\alpha\beta} \approx \begin{bmatrix} 0 & -\frac{1}{2(q_1^2+q_2^2)} \\ \frac{1}{2(q_1^2+q_2^2)} & 0 \end{bmatrix} \tag{2.121}$$

Starting from the set:

$$\{q_1, q_2\} = 0 \tag{2.122}$$

$$\{p_1, p_2\} = 0 \tag{2.123}$$

$$\{q_1, p_1\} = 1 \tag{2.124}$$

$$\{q_2, p_2\} = 1 \tag{2.125}$$

$$\{q_1, p_2\} = 0 \tag{2.126}$$

$$\{q_2, p_1\} = 0 \tag{2.127}$$

We get:

$$\{q_1, q_2\}_{DB} = 0 \tag{2.128}$$

$$\{p_1, p_2\}_{DB} = -\frac{1}{r^2}(q_1 p_2 - q_2 p_1) \tag{2.129}$$

$$\{q_1, p_1\}_{DB} = 1 - \frac{q_1^2}{r^2} \tag{2.130}$$

$$\{q_2, p_2\}_{DB} = 1 - \frac{q_2^2}{r^2} \tag{2.131}$$

$$\{q_1, p_2\}_{DB} = -\frac{q_1 q_2}{r^2} \tag{2.132}$$

$$\{q_2, p_1\}_{DB} = -\frac{q_1 q_2}{r^2} \tag{2.133}$$

For instance the equation of motion in the Dirac prescription became:

$$\dot{q_1} = \{q_1, H\}_{DB} = \frac{p_1}{m} \tag{2.134}$$

$$\dot{q_2} = \{q_2, H\}_{DB} = \frac{p_2}{m} \tag{2.135}$$

$$\dot{p_1} = \{p_1, H\}_{DB} = \frac{p_1^2 + p_2^2}{m} q_1 \tag{2.136}$$

$$\dot{p_2} = \{p_2, H\}_{DB} = \frac{p_1^2 + p_2^2}{m} q_2 \tag{2.137}$$

Sometimes its easier to work in a system with constraints on the constrained coordinate instead of the unconstrained one. Sometimes it is not possible to find an unconstrained coordinate at all. All the methods so far work fine in classical physics, let's translate this in the next lecture to quantum physics and see how to deal with the constraints in a systematic manner.

3 Lecture III.

3.1 Quantization

Now that we have a better grasp of classical mechanics and how to deal with constraints in it lets try to quantize a classical theory. Here we end up in a problem really fast because of the usual canonical quantization:

$$\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}] \tag{3.1}$$

Does not exist in general. This problem is related to the fact that quantum mechanics is more fundamental than classical mechanics. Because of that, it is impossible to describe all quantum mechanics trough deformations of classical physics and such a map cannot exist. The way to deal with that is to use such deformation quantization and get a family of quantum theory that has the same classical limit, then see if any of them make sense and what are the differences between them. Of course in such a method it is impossible to arrive at a fundamentally quantum theory with no classical analog, for such a thing there are other methods like S matrix analysis. Just to illustrate the impossibility of such a quantization map one would want such a map $\mathcal{Q}$ that act on phase space functions $g(q,p)$ in such a way that:

- $\mathcal{Q}(g(q,p)) = \mathcal{Q}_{g(\hat{q},\hat{p})}$
- $\mathcal{Q}_q \left|\psi\right\rangle = q \left|\psi\right\rangle$
- $\mathcal{Q}_p \left|\psi\right\rangle = -i\hbar\frac{\partial}{\partial q} \left|\psi\right\rangle$
- $g \to \mathcal{Q}_g$ is linear
- $[\mathcal{Q}_A, \mathcal{Q}_B] = i\hbar\mathcal{Q}_{\{A,B\}}$
- $\mathcal{Q}_{A(B)} = A(\mathcal{Q}_B)$

Not only are these four properties mutually inconsistent, but any three of them are also inconsistent. For instance, if we accept the usual form for the position operator, linearity and the commutator relation being of order $\hbar$ only asymptotically($\hbar \to 0$), leads to deformation quantization. The non-existence of a true map to quantize phase space functions gives us the problem of ordering when going to quantum mechanics. This order ambiguity is the fact that may exist multiple operators in Hilbert space that represent a given phase space function in the classical limit. This means the quantum Hamiltonian isn't uniquely determined by the classical limit, as one would expect. This ambiguity rises even when we demand the operators to be Hermitian. This is not strange if one thinks that the quantum Hamiltonian has all the information about the theory, including its classical limit. This ordering ambiguity arises in quantum field theory as well. In that particular case, all the ambiguity that arises is actually UV divergent. The way to surpass this problem is actually rephrase what we want, we don't want a deformation of classical physics that is consistent with quantum mechanics. We want a consistent quantum theory that could have a classical limit. From now one we will ignore the operator ordering issue[6] and assume a given quantum theory that is uniquely determined and maybe have others with a similar classical limit. This will not be a problem in the parameterized particle but in general, is something that one should need to fix (we will see that in the Bosonic String case). Now, in an unconstrained system, the application of such quantization is fixed, we create a Hilbert space and transform the phase space functions in operators(choosing an ordering). The Hamiltonian is transformed in the operator of time evolution and its operator equation gives the Schrodinger equation, all the operator equations are analog to the classical counterpart. The time evolution is then:

$$\dot{A} = \frac{1}{i\hbar}[A, H] \tag{3.2}$$

As operators, this should be acted on states. Let's see what would change when we add constraints, assuming a set of second-class constraints C_α, one should then impose in the states:

$$\hat{C}_\alpha \left|\psi\right\rangle = 0 \tag{3.3}$$

But notice that this would imply that:

$$[\hat{C}_\alpha, \hat{C}_\beta] \left|\psi\right\rangle = 0 \tag{3.4}$$

This should hold for all $\left|\psi\right\rangle$, trough the canonical quantization procedure this would mean that the classical counterpart should follow:

$$\{C_\alpha, C_\beta\} = 0 \tag{3.5}$$

However, because these are second class constraints they don't even vanish weakly, and this is impossible to enforce such constraint. We need to be more careful when quantizing constrained systems, let's go through the steps to make it in a consistent manner.

3.2 The road to the quantum realm.

Continuing to work with a second class constraint system we can remember the Dirac bracket formalism where all the constraint are enforced and we can deal only with strong equality. In such a bracket we indeed have:

$$\{C_\alpha, C_\beta\}_{DB} = 0 \tag{3.6}$$

Than this gives us a way to quantize our theory, the quantization map should be:

$$\{A, B\}_{DB} \to \frac{1}{i\hbar}[A, B] \tag{3.7}$$

This quantization scheme has the feature that the commutator of any phase space function with the constraints are zero. Now to the interesting part, any operator that

[6] This ambiguity appears in quantum field theory in the form of ambiguities in the renormalization schemes.

commutes with every conceivable function of $\hat{q}$ and $\hat{p}$ cannot itself be a function of the phase space operators. The only possibility is that the operator acts as a c-number in the states:

$$\hat{C}_\alpha |\psi\rangle = \lambda_{c_\alpha} |\psi\rangle \tag{3.8}$$

To satisfy the constraint we just need to assume that this eigenvalue is zero, this means that:

$$\hat{C}_\alpha |\psi\rangle = 0 \tag{3.9}$$

Is not a constraint in the phase space but a operator identity:

$$\hat{C}_\alpha = 0 \tag{3.10}$$

Doing the quantization with Dirac brackets, the second class constraints can be freely set to zero in any state (of course the constraint information is inside the Dirac bracket). Now one can easily quantize a theory with only second class constraints, it is just a matter of finding the Dirac bracket and then doing the quantization map. While the procedure is simply stated, it is not so easily implemented.

In order to visualize this process we can revisit the second class constraint system that we talked about:

$$q^1 \approx 0 \tag{3.11}$$

$$p_1 \approx 0 \tag{3.12}$$

We can see that going to quantum mechanics, the use of Poisson bracket would give the wrong result:

$$[\hat{q}^1, \hat{p}_1] = i\hbar \tag{3.13}$$

This acting on physical states would be inconsistent with:

$$\hat{q}^1 = 0 \tag{3.14}$$

$$\hat{p}_1 = 0 \tag{3.15}$$

But the Dirac Bracket would give the right answer:

$$[\hat{q}^1, \hat{p}_1] = i\hbar\{\hat{q}^1, \hat{p}_1\}_{DB} \tag{3.16}$$

In this case, there is no order ambiguity and quantization proceeds in a smooth way. One way to justify this process of quantizing using Dirac bracket is to solve classically the constraint:

$$C_\alpha^2 = 0 \tag{3.17}$$

For some variables z^α, in terms of the rest of the variables x^i:

$$z^\alpha = z^\alpha(x^i) \tag{3.18}$$

The bracket between the remaining variables would depend only in x^i:

$$\{x^i, x^j\} = F^{ij}(x) \tag{3.19}$$

This matrix being invertible. The quantum theory is defining by finding a representation of this algebra for the operator $\hat{x}^i$:

$$[x^i, x^j] = i\hbar F^{ij}(\hat{x}) \tag{3.20}$$

Once a solution is found we can go the other way and define the operator $\hat{z}^\alpha$ as function of $\hat{x}$ with a fix ordering. This implies that:

$$\hat{C}_\alpha^2 = 0 \tag{3.21}$$

We can see that to this quantization to work we need to find an explicit representation of the Dirac Bracket (in this case the bracket of the remaining variables is the Dirac bracket). This is a big problem with no general solution except for easier cases. In general, it is impossible to find an explicit solution, even if we let high order corrections take place:

$$[\hat{A}, \hat{B}] = i\hbar\{A, B\} + \mathcal{O}(\hbar^2) \tag{3.22}$$

This is the main reason as to why see a second class constraint as a gauge fixed first class constraint. Now we need to know how to deal with first class constraints. We will work out some ways to handle first class constraint. The first one is the reduced phase space that tries to quantize only gauge invariant objects. The second is the Dirac Procedure to quantize first class constraints that are more powerful and has a nice physical interpretation. The most advanced techniques revolve around generalizing the Dirac Procedure so in this lecture we will work out more the Dirac way to have a better understanding of the first class case. If you grasp the Dirac quantization the BRST is just a clever way to deal with all the choices that one has to do in the quantization procedure. Working with the Dirac Procedure will give us a better intuition when dealing with gauge theories that we have in Physics.

3.3 Reduced phase space quantization.

Let's consider only first class constraint. In this case, it is possible to try to get rid of the redundancy early one. It is possible to archive that if we only quantize gauge invariant functions. This method is called reduced phase space quantization. To proceed with this method it is necessary to have a complete set of gauge invariant functions that describe the classical system. With this set, we try to find the quantum space as an irreducible representation of the commutation relations of these functions. Every state in this Hilbert space would be physical by construction. One example in the case that we saw early one would be:

$$L = \frac{m}{2}(\dot{q}^1 + \dot{q}^2)^2 - V(q^1 + q^2) \tag{3.23}$$

We worked out the complete set of gauge invariant functions as:

$$q_{cm} = q^1 + q^2 \tag{3.24}$$

$$p_{cm} = p_1 + p_2 \tag{3.25}$$

In the Schrodinger representation we would have:

$$\hat{q}_{cm} |\Psi\rangle = q_{cm} |\Psi\rangle \tag{3.26}$$

$$\hat{p}_{cm} |\Psi\rangle = -i\hbar \frac{\partial}{\partial q_{cm}} |\Psi\rangle \tag{3.27}$$

There is no gauge dependent objects in this space, gauge invariance is manifest. In practice it is very hard to find a complete set of observable as this is equal to solve:

$$\{F(q,p), C^1_m\} \approx 0 \tag{3.28}$$

It is possible to reach the reduced phase space by a different approach that only works when there is no Gribov problem. It consists on fixing the gauge classically:

$$G_m = 0 \tag{3.29}$$

This works because any function can be view after the gauge fixing as the restriction in that gauge of a gauge invariant function. After the gauge fixing, one is effectively working with gauge invariant function. A complete set of independent gauge fixed functions provides one with a complete set of gauge invariant objects. Now the quantization is identical to a second class constraint one.

Both quantizations seem natural since we work only with gauge invariant objects. The Hilbert space constructed has only physical states because of this. However, this approach can be difficult to implement, finding a complete set of observable may be hard and even spoil manifest invariance under some symmetry. In the case of field theory, elimination of gauge degrees of freedom generally destroys locality. Finally, the bracket of the observables could be impossible to render in a quantum mechanical setup. Because of that, it is important to construct an alternative method of quantization of first class constraint. We will work out only the Dirac method for such a system.

3.4 Dirac quantization procedure.

Similarly with the second class constraint quantization, when we have a set of first and second class constraint $C_\alpha = (C_i^2, C_a^1)$ we should define the Dirac bracket in such a way that:

$$\{C_i^2, f(q,p)\}_{DB} = 0 \tag{3.30}$$

$$\{C_a^1, C_b^1\}_{DB} = f_{abc} C_c^1 \tag{3.31}$$

The Dirac quantization process just like before, we use the quantization map to define our commutator, ignoring the ordering, and the second class constraint vanishes as operator identities. Having dealt with the second class constraint we can ignore them and focus on the first class one. Now we can't do the same thing with first class constraint and this means that the operator equation became a constraint in Hilbert space:

$$C_a^1 \left|\Psi\right\rangle = 0 \tag{3.32}$$

This states that we started with a nonphysical Hilbert space, called kinematical Hilbert space, that had more degree of freedom than the physical one. Not all states in this space are physical and to construct the physical Hilbert space we demand that they obey the equation (3.32). Once we construct the physical Hilbert space we need to find the physical inner product in this space, something that is not always possible. Dirac quantization procedure for systems with first class constraints: first quantize using Dirac brackets and then restrict the Hilbert space by demanding that constraint operators annihilate physical states. Lastly, we need to define observables that are gauge invariant. On a quantum level, we would like to be able to only work with operators that map physical states into physical states. If this were not the case, an operator would have no eigenbasis in the physical state space and we could not even define its expectation value. At the classical level, this implies that the classical quantities equivalent to quantum observables are first class. But we saw that first-class quantities are gauge-invariant in the Hamiltonian formalism. Therefore, quantum observables in the Dirac quantization scheme correspond to classical gauge invariant quantities. It seems as if our reduction of the original Hilbert space has somehow removed the gauge degrees of freedom from our system, provided that we only work with operators whose domain is the physical Hilbert subspace. Indeed, when such a choice is made, the arbitrary operators that play the role of Lagrange multiplier in the operator for the total Hamiltonian have no role and the time evolution of $\left|\Psi\right\rangle$ is determined conclusively:

$$\hat{H}_T \approx \hat{H} + \hat{\lambda}_a \hat{C}_a^1 \tag{3.33}$$

$$\hat{H}_T \left|\Psi\right\rangle = \hat{H} \left|\Psi\right\rangle \tag{3.34}$$

This is it and the theory is quantized. Just to remind a highly non trivial thing, the ordering product makes that when going to quantum mechanics we actually have something like:

$$[\hat{C}^1_a, \hat{C}^1_b] = i\hbar f_{abc}\hat{C}^1_c + \mathcal{O}(\hbar^2) \tag{3.35}$$

$$[\hat{C}^1_a, H] = i\hbar g_{ab}\hat{C}^1_b + \mathcal{O}(\hbar^2) \tag{3.36}$$

And for physical states:

$$[\hat{C}^1_a, \hat{\mathcal{O}}] = i\hbar k_{ab}\hat{C}^1_b + \mathcal{O}(\hbar^2) \tag{3.37}$$

The fact that we ignore the ordering problem is related to retaining only the liner order in $\hbar$, which in general is not good. One would want to find relations that are only linear in $\hbar$. There is no guarantee that such a thing is possible, especially when we try to satisfy the last condition for every classical gauge-invariant quantity. This is, for example, a highly non−trivial problem in loop quantum gravity. Now someone could ask whether or not this quantization procedure is equivalent to the gauge fixed one. Until now there is no general proof that Dirac quantization is equivalent to canonical quantization for first class systems (In all physical cases so far this is true). Again we are faced with a choice that represents another ambiguity in the quantization procedure.

3.5 Quantum Anomaly

Before we proceed to some basic examples lets see what could go wrong in the Dirac quantization program. To the quantization to work out we assumed that the canonical bracket is preserved in quantum mechanics:

$$[\hat{C}_m, \hat{C}_n] = i\hbar f^p_{mn}\hat{C}_p \tag{3.38}$$

Such that:

$$\hat{C}_m \left|\Psi\right\rangle = 0 \tag{3.39}$$

Is consistent with:

$$[\hat{C}_m, \hat{C}_n] \left|\Psi\right\rangle = 0 \tag{3.40}$$

However it is possible to exist higher order corrections that arises from quantum mechanics:

$$[\hat{C}_m, \hat{C}_n] = i\hbar f^p_{mn}\hat{C}_p + \hbar^2\hat{A}_{mn} \tag{3.41}$$

In this case, to be possible to proceed and the theory be consistent we need:

$$\hat{A}_{mn} \left|\Psi\right\rangle = 0 \tag{3.42}$$

This condition has no classical counterpart and restricts to much the physical space, normally it implies that:

$$\left|\Psi\right\rangle = 0 \tag{3.43}$$

So it is not possible to impose that this term annihilates the physical states. This means that the picture has changed, the quantum operator $\hat{C}$ is no longer first class and can't be interpreted as gauge generators. The gauge invariance is broken at the quantum level and the term $\hat{A}_{mn}$ is called gauge anomaly. If the gauge invariance is broken it is meaningless to apply the Dirac quantization. The anomaly in the gauge invariance could appear as well dynamically in the commutator with the Hamiltonian:

$$[\hat{H}_0, \hat{C}_m] = i\hbar d_m^n \hat{C}_n + \hbar^2 B_m \tag{3.44}$$

This means that the time evolution is not gauge invariant and would be possible to go from a physical state to a nonphysical one. Gauge invariance is broken again. In the Dirac procedure it is almost impossible to deal with these anomalous terms and in most of the cases that we will work, they will be zero. It turns out that sometimes we can be clever in the definition of physical state such that even that an anomalous term appears we can still have gauge invariance. We will have to deal with that in the case of Bosonic String, this method is called Fock quantization but we will not proceed any further with that. This comes automatically in the BRST quantization as sometimes it is consistent even with an anomaly.

3.6 Quantization of the first example: Gauge redundancy

Lets see how the quantization procedure runs for the case:

$$\mathcal{L} = \frac{m}{2}(\dot{q}_1 + \dot{q}_2)^2 - V(q_1 + q_2) \tag{3.45}$$

We know how to quantize the gauge fixed action as is just a free shifted particle, the interesting thing is to do the Dirac procedure in a gauge invariant way. The constraint is unique and we start with the kinematical Hilbert space

$$\mathcal{H}_{kin} = \{|\Psi\rangle\} = \mathcal{L}^2(\mathbb{R}^2) \tag{3.46}$$

And with the usual inner product:

$$\langle\psi|\phi\rangle = \int \mathrm{d}q_1\, \mathrm{d}q_2\, \bar{\psi}(q_1, q_2)\phi(q_1, q_2) \tag{3.47}$$

The constraint need to be transformed in an operator in this space, this is done without ambiguity because of the linearity:

$$\hat{C} = \hat{p}_2 - \hat{p}_1 \tag{3.48}$$

The next step in the Dirac quantization procedure is to construct the physical space, as the space of solutions of:

$$(\hat{p}_2 - \hat{p}_1)\,|\Psi\rangle = 0 \tag{3.49}$$

It is easy to see that in position representation the physical wave function will satisfy:

$$\frac{\partial\Psi}{\partial q_2} - \frac{\partial\Psi}{\partial q_1} = 0 \tag{3.50}$$

Then the physical state will be formed by wave functions of the form:

$$\Psi(q_1 + q_2) \tag{3.51}$$

And the physical Hilbert space is then:

$$\mathcal{H}_{phy} = \{|\Psi\rangle\} = \mathcal{L}^2(\mathbb{R}) \tag{3.52}$$

In this case, it is easy to see that the kinematical inner product is infinity for physical states. It is necessary to find another inner product in this space, in this case, this is a trivial task and a simple integration on only one variable do the trick. Another remark is that the physical wave function is gauge invariant. During the classical description, we needed to solve the constraint and then consider equivalence classes of states generated by the gauge symmetry to find gauge-invariant objects. Now, in quantum mechanics, the act of solving the constraint already give us a gauge invariant object

3.7 Quantization of Proca Field.

This point now let's apply this knowledge to quantum field theory. The procedure is similar only going from Hilbert to Fock space and the operators being distributions. The Proca field is a good example of a second class constraint that can be quantized very easily, it describes a massive vector field. The Lagrangian density is such that:

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{m^2}{2}B_\mu B^\mu \tag{3.53}$$

With $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The conjugated momenta that we need to go to Hamiltonian formalism is:

$$\Pi_\mu = -F_{0\mu} \tag{3.54}$$

And we can see that we have the constraint:

$$\Pi_0 = 0 \tag{3.55}$$

The usual canonical commutation is set:

$$\{B_\mu(x), \Pi_\nu(y)\} = \eta_{\mu\nu}\delta^3(x-y) \tag{3.56}$$

Writing the Hamiltonian we have:

$$H = \int \mathrm{d}^3x \, \frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}B_{ij}B_{ij} - B_0\partial_i\Pi_i - \frac{m^2}{2}B_0B_0 + \frac{m^2}{2}B_iB_i + \lambda_1\Pi_0 \tag{3.57}$$

The consistency condition of the constraint trough time evolution is:

$$\{\Pi_0(x), H\} = \int \mathrm{d}^3z \, \{\Pi_0(x), -B_0\partial_i\Pi_i\} - \frac{m^2}{2}\{\Pi_0, B_0B_0\} = -\partial_i\Pi_i + m^2B_0 \tag{3.58}$$

If we demand that this constraint is preserved trough time evolution we get to use this as a constraint, being identified as the equation of motion of B_0:

$$\partial_i \Pi_i - m^2 B_0 = 0 \tag{3.59}$$

Then the Hamiltonian is:

$$H = \int \mathrm{d}^3x \, \frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}B_{ij}B_{ij} - B_0\partial_i\Pi_i - \frac{m^2}{2}B_0B_0 + \frac{m^2}{2}B_iB_i + \lambda_1\Pi_0 + \lambda_2(\partial_i\Pi_i - m^2B_0) \tag{3.60}$$

This is the last constraint as the time evolution of it does not generate any new constraint, only fixes the Lagrange multiplier. Because we will use the Dirac method we do not need to evaluate what is the Lagrange multiplier as this information will be in the Dirac bracket. We can find out directly that they are second class and has the matrix of Poisson bracket between then as:

$$\{C_i, C_j\} = \begin{bmatrix} 0 & m^2\delta^3(x-y) \\ -m^2\delta^3(x-y) & 0 \end{bmatrix} \tag{3.61}$$

This matrix is non singular because all the constraint are second class and we can find its inverse:

$$\{C_i, C_j\}^{-1} = \begin{bmatrix} 0 & -\frac{1}{m^2}\delta^3(x-y) \\ \frac{1}{m^2}\delta^3(x-y) & 0 \end{bmatrix} \tag{3.62}$$

With that we can construct the Dirac brackets of any phase space function and for our case, starting with the equal times Poisson brackets:

$$\{B^\mu(x), B^\nu(y)\} = 0 \tag{3.63}$$

$$\{\Pi_\mu(x), \Pi_\nu(y)\} = 0 \tag{3.64}$$

$$\{B^\mu(x), \Pi)\nu(y)\} = \delta^3(x-y)\delta^\mu_\nu \tag{3.65}$$

We arrive at the Dirac brackets:

$$\{B^i(x), B^0(y)\}_{DB} = \frac{1}{m^2}\partial^i\delta^3(x-y) \tag{3.66}$$

$$\{B^i(x), \Pi_j(y)\}_{DB} = \delta^3(x-y)\delta^i_j \tag{3.67}$$

$$\{\Pi_\mu(x), \Pi_\nu(y)\}_{DB} = 0 \tag{3.68}$$

$$\{B^0(x), \Pi_j(y)\}_{DB} = 0 \tag{3.69}$$

$$\{B^\mu(x), \Pi_0(y)\}_{DB} = 0 \tag{3.70}$$

Now to quantize this theory we promote the Dirac bracket to the commutator and the constraints as operator identities. This is a trivial task and we are done, the Proca field is quantized and the Fock space is trivially constructed as there are only physical degrees of freedom and we used all the constraints. It is possible to check that this is the right commutation relations for the field using the constraint to compute $[B^0, \Pi_0]$

4 Lecture IV.

4.1 Application of Dirac Quantization Procedure

Now that we understand better how to deal with constraints in the Hamiltonian formalism and how to quantize a theory with constraints lets apply this Dirac Method to more difficult and useful problems. The set of theories that we will work first are reparametrization invariant theories. We will study the free particle(non-relativistic and relativistic case) and closed bosonic string. With the knowledge acquired from these theories, one can start to tackle more hard problems like canonical quantization of Gravity or even go deeper in String Theory. Let's review a little about reparametrization invariance in the classical setup and then jump into the different cases. In this lecture, we will work only in the flat space-time, but the generalization for other space-times is straightforward.

4.2 Reparametrization Invariance

In general, a theory invariant for reparametrization has the following symmetry:

$$S = \int \mathrm{d}\tau \, L(q^i(\tau), \frac{dq}{d\tau}, \tau) = \int \mathrm{d}\tau' \, \lambda L(q(\tau'), \frac{dq}{d\tau'}, \tau') \tag{4.1}$$

With λ an arbitrary real number. This is not the most general case as one could have this symmetry up to a total derivative, but let's work in this section assuming that the total derivative term is zero. This theory is invariant by the transformation of the evolution parameter τ:

$$\tau' = f(\tau) \tag{4.2}$$

We can find a general form for a Lagrangian to be invariant under this symmetry, this let us transform any non-singular theory into a singular theory as this symmetry has the form of a gauge symmetry. Starting from an action that has the physical time as evolution parameter:

$$S = \int \mathrm{d}t \, L(q, \dot{q}, t) \tag{4.3}$$

We can introduce the reparametrization invariance in the following form:

$$L_R = L(q, \frac{\dot{q}}{\dot{\tau}}, \tau)\dot{\tau} \tag{4.4}$$

$$S = \int \mathrm{d}\tau \, L_R(q, \dot{q}, t, \dot{t}, \tau) \tag{4.5}$$

The physical time became a dynamical variable and the evolution parameter now is a nonphysical time-like parameter. Before we investigate the Hamiltonian formulation of such a theory we can ask why one would introduce such symmetry. The first motivation is to preserve some symmetry of the problem, as will be the case of relativistic particle and string theory. Another motivation is to understand Gravity better because reparametrization are just diffeomorphism of the real line and the gauge group of Gravity can be considered as the diffeomorphism group of space-time. This procedure could simplify our life as we could transform a Hamiltonian that depends on the evolution parameter explicitly into one that does not have this dependence but has additional degrees of freedom. Because in the Dirac program we need the action to be written as an integral of a Lagrangian over time, we have the problem that some theories don't have the action given naturally in this form. For the relativistic particle or string, it is given as an integral over the world-line or world-sheet instead. To pass this problem we consider some time-like parameter which behaves like the usual time coordinate and quantizes using this time. We can forget that we did this procedure as this information will be stored in the theory through the gauge symmetries. Let see how we could, in general, go to the Hamiltonian formalism using the reparametrization invariant action (4.5):

$$p_i = \frac{\partial L_R}{\partial \dot{q}^i} \tag{4.6}$$

$$p_0 = \frac{\partial L_R}{\partial \dot{t}} \tag{4.7}$$

Using the definition of L_R we can find:

$$p_0 = L - \frac{\dot{q}^i}{\dot{t}}\frac{\partial L}{\partial \dot{q}^i} + \dot{t}\frac{\partial L}{\partial \dot{t}} \tag{4.8}$$

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{4.9}$$

We can introduce a covariant formulation by defining:

$$x^\mu = (t, q^i) \tag{4.10}$$

Such that the Hamiltonian for the unconstrained system appear:

$$p_0 = -H(x^\mu, \pi_\mu)|_{\dot{q}\to\frac{\dot{q}}{\dot{t}}} \tag{4.11}$$

This equation appear as a constraint, because we can't write $\dot{t}$ as a function of the other coordinates. If in the non-singular theory we had this inversion:

$$\dot{q}^i = f(q^i, p_i, \tau) \tag{4.12}$$

Then in this system we only have the inversion of the space components like:

$$\dot{q}^i = \dot{t} f(q^i, p_i, t) \tag{4.13}$$

We would not be able to invert $\dot{t}$ as functions of the phase space variables. The Hamiltonian in the reparametrization invariant theory will be:

$$H_R = p_0\dot{t} + p_i\dot{q}^i - L_R = \dot{t}\left(p_0 - L + p_i\frac{\dot{q}^i}{\dot{t}}\right) = \tag{4.14}$$

$$= \dot{t}(p_0 + H)|_{\dot{q}\to\frac{\dot{q}}{\dot{t}}}$$

Calling the constraint C_1 and identifying the $\dot{t}$ as the Lagrange multiplier we can see that the Hamiltonian is actually zero in the constraint surface:

$$C_1 = p_0 + H(x^\mu, \pi_\mu) \tag{4.15}$$

$$H_R \approx \lambda(t)C_1 \tag{4.16}$$

This means that we don't have any evolution! This is to be expected as the time parameter is nonphysical and one should look for the evolution with respect to the physical time coordinate. This is a general feature of diffeomorphism invariant theories (There are cases where this is not true and the Hamiltonian does not vanish). This means that all the time evolution are gauge transformations and aren't physical, this is something that we will have to deal when going to the quantum description of such a system. Now, let's specialize deeper in the free particle case to get a better grasp of the Dirac procedure in theories like this.

4.3 Non-Relativistic Free particle with reparametrization invariance: Dirac Quantization

Starting from the action for a non−relativistic free particle:

$$S = \int \mathrm{d}t\, \frac{\dot{q}^2}{2m} \tag{4.17}$$

We can introduce a gauge symmetry(first class constraint) by expanding the coordinate space:

$$S = \int \mathrm{d}s\, \frac{q''^2}{2mt'} \tag{4.18}$$

Where the prime derivative means:

$$t' = \frac{dt}{ds} \tag{4.19}$$

This action is invariant under reparametrizations on s, going to the Hamiltonian description the canonical momenta is:

$$p_t = -m\frac{q'^2}{t'^2} \tag{4.20}$$

$$p_q = m\frac{q'}{t'} \tag{4.21}$$

It is clear that we have a constraint in this system:

$$C = p_t + \frac{p_q^2}{2m} = 0 \tag{4.22}$$

Now the Hamiltonian in this case is:

$$H_T \approx t'(p_t + \frac{p_q^2}{2m}) \tag{4.23}$$

This means that the Hamiltonian is zero on the constraint surface, systems of this kind are called totally constrained systems. Because t' plays the role of a Lagrange multiplier we can write:

$$H_T \approx \lambda(s)C \tag{4.24}$$

This is the only constraint and by consequence this is a first class constraint:

$$\{C, C\} = 0 \tag{4.25}$$

$$\{C, H\} = 0 \tag{4.26}$$

Lets see the flow generated by H_T:

$$q' = \{q, H_T\} = \lambda \frac{p_q}{m} \tag{4.27}$$

$$t' = \{t, H_T\} = \lambda \tag{4.28}$$

$$p_q' = \{p_q, H_T\} = 0 \tag{4.29}$$

$$p_t' = \{p_t, H_T\} = 0 \tag{4.30}$$

It is clear to see that in (4.27) starting from the same initial condition we can have different solutions only changing the λ parameter. This means that the information of q and t as a function of s is not important. The important question that one should ask is relational questions like: What is the value of q when t is something? In some sense fixing the gauge in this system is choosing a specific value for one of theses functions and the relational question became fundamental. We can look for the gauge orbits in this case very easily, the gauge transformations are:

$$\frac{dq}{d\chi} = \{q, C\} = \frac{p_q}{m} \tag{4.31}$$

$$\frac{dt}{d\chi} = \{t, C\} = 1 \tag{4.32}$$

$$\frac{dp_q}{d\chi} = \{p_q, C\} = 0 \tag{4.33}$$

$$\frac{dp_t}{d\chi} = \{p_t, C\} = 0 \tag{4.34}$$

The gauge orbits are:

$$q(\chi) = q + \frac{p_q}{m}\chi \tag{4.35}$$

$$t(\chi) = t + \chi \tag{4.36}$$

$$p_q(\chi) = Cte = p_0 \tag{4.37}$$

$$p_t(\chi) = -\frac{p_q^2}{2m} \tag{4.38}$$

In the last one, the constraint is used. This defines the flow on the constraint surface. Now as stated in the Dirac procedure we need to look for Dirac observables. We need to look for functions on phase space that are constant on the gauge orbit (gauge invariant). Let F be a Dirac observable, then:

$$\{F, C\} \approx 0 \tag{4.39}$$

Because the kinematical phase space has 4 degree of freedom and the gauge invariance kills two of them we expect 2 linearly independent Dirac observables to describe our system. This process is not always straight forward if you choose a bad gauge fixing. But for the sake of illustration lets do the easier route. In our case because there is only one constraint we need only one gauge fixing, let's choose:

$$t(s) = \tau; \tau \in \mathbb{R} \tag{4.40}$$

This fixes one point on the gauge orbit of t to fix the value of q. Now given a phase space function f we associate a physical observable $F(\tau)$ associating the value f(s) for all point in the gauge orbit G(s) trough s. Specifically, the observable associated to q, $F_q(\tau)$ is the value of q(s) at the point of the gauge orbit where t(s) takes the value τ. In this construction we get:

$$F_q = q + \frac{p_q}{m}(\tau - t) \tag{4.41}$$

The second Dirac observable is related to the momentum, which is constant:

$$F_p = p_q \tag{4.42}$$

All other gauge invariant observable are linear combinations of theses 2. Now the next step is to quantize this system. Following our procedure we have only one first class constraint, the kinematical Hilbert space is then constructed:

$$\mathcal{H}_{kin} = \{|\psi\rangle\} = \mathcal{L}^2(\mathbb{R}^2) \tag{4.43}$$

Where the phase space is 4 dimensional:

$$\hat{q}\,|\psi\rangle = q\,|\psi\rangle \tag{4.44}$$

$$\hat{t}\,|\psi\rangle = t\,|\psi\rangle \tag{4.45}$$

$$\hat{p}_q\,|\psi\rangle = -i\hbar\frac{\partial}{\partial q}\,|\psi\rangle \tag{4.46}$$

$$\hat{p}_t\,|\psi\rangle = -i\hbar\frac{\partial}{\partial t}\,|\psi\rangle \tag{4.47}$$

With a kinematic inner product in this Hilbert space:

$$\langle\psi|\phi\rangle = \int \mathrm{d}q\,\mathrm{d}t\,\bar{\psi}(q,t)\phi(q,t) \tag{4.48}$$

And now we transform our constraint in a hermitian operator in this Hilbert space:

$$\hat{C} = -i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} \tag{4.49}$$

In this case there is no problem in ordering and we can proceed without any problems. The next step in the Dirac program is to find the space of solutions of the constraint equation that will define the physical Hilbert space. It is clear to see that the constraint equation is just the usual Schrodinger equation:

$$\hat{C}\,|\psi\rangle = \left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial}{\partial q}\right)|\psi\rangle = 0 \tag{4.50}$$

We can write the solution of the constraint equation as:

$$|\Psi(q,t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_{free}}\,|\Psi(q,0)\rangle \tag{4.51}$$

A immediate thing that we can see is that the kinematical inner product is infinite with physical states:

$$\langle\Psi|\Phi\rangle = \int \mathrm{d}q\,\mathrm{d}t\,\bar{\psi}(q)\phi(q) \tag{4.52}$$

This is not a good inner product to the physical Hilbert space. This is a normal feature of non-compact gauge orbits(the integral in time diverges because of the non-compactness). In this particular case its easy to find the physical Hilbert space, this is not a straightforward thing to do in general. Because we know the solution of the unparameterized particle:

$$\langle\Psi|\Phi\rangle = \int \mathrm{d}q\,\bar{\psi}(q)\phi(q) \tag{4.53}$$

This is the physical inner product in the physical Hilbert space defined by the constraint. The physical Hilbert space is then identified as:

$$\mathcal{H}_{phy} = \{|\Psi\rangle\} = \mathcal{L}^2(\mathbb{R}) \tag{4.54}$$

Then we have identified the correct Hilbert space and all the procedure goes as usual. The last step is to find the gauge invariant observables, this is straightforward and the theory is quantized:

$$\hat{F}_q = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0) \tag{4.55}$$

$$\hat{F}_p = \hat{p}_q \tag{4.56}$$

And Indeed:

$$[\hat{F}_q, \hat{C}] = 0 \tag{4.57}$$

$$[\hat{F}_p, \hat{C}] = 0 \tag{4.58}$$

As is expected.

4.4 Relativistic Particle

The relativistic particle is interesting because falls into the category of actions that aren't in the standard form that Dirac quantization needs, the action for such particle is given by the integral of the proper time:

$$S = -m \int \mathrm{d}\tau \tag{4.59}$$

Such that:

$$\mathrm{d}\tau^2 = \mathrm{d}x^{0^2} - \mathrm{d}x^{i^2} \tag{4.60}$$

It is useful in this case to analyze the particle coupled to an electromagnetic field so one can see how the Dirac quantization procedure would go in the interaction case, in the end, we will finish only the free case for illustration. The action writing in terms of the time like coordinate for the free case is:

$$S = -m \int \mathrm{d}\gamma \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{4.61}$$

Where the dot is the derivative with respect to the time like parameter γ. If we want to couple this to a electromagnetic field we just add the Lorentz term:

$$S = \int \mathrm{d}\gamma \left(-m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - q\dot{x}^\mu A_\mu\right) \tag{4.62}$$

First we note that in both cases the massless limit is not clear, this can be solved by adding a additional field, however we will go to the Hamiltonian formalism soon and there it will became clear how one could go to this limit if wanted. The first step for going to the Hamiltonian formalism is to find the canonical momenta:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} - eA_\mu \tag{4.63}$$

We can see that we have the primary constraint:

$$C_1 = \frac{1}{2}((p_\mu + eA_\mu)(p^\mu + eA^\mu) + m^2) \tag{4.64}$$

Now we can construct the Hamiltonian of this theory:

$$H = p_\mu \dot{x}^\mu - L = 0 \tag{4.65}$$

This is what we expected for a reparametrization invariant theory, extending the action we write this as:

$$H_{ext} \approx \lambda(\gamma) C_1 \tag{4.66}$$

Now it is clear that there is only one constraint and it is first class. In this formalism the massless limit is trivial to do and the constraint in this case would be:

$$C_1 = \frac{1}{2}(p_\mu + eA_\mu)(p^\mu + eA^\mu) \tag{4.67}$$

Now we will set the electromagnetic field to zero and proceed with the quantization. When we have finished the Dirac procedure we will come back with the A_μ to see what would change. Now the constraint reads:

$$C_1 = (p_\mu p^\mu + m^2) \tag{4.68}$$

This is just the mass shell condition for the particle, it appear as a constraint in our theory. We can analyze the gauge transformations generated by this constraint:

$$\delta x^\mu = \epsilon(\gamma)\{x^\mu, C_1\} = 2\epsilon p^\mu \tag{4.69}$$

$$\delta p_\mu = \epsilon(\gamma)\{p_\mu, C_1\} = 0 \tag{4.70}$$

This is a displacement in the variable x^μ along the direction of p_μ. This means that if we make a gauge transformation, the particle will appear somewhere else in space-time with the same momentum. If we look at the class of all states that are equivalent to some given state, we find that it is a straight line in phase space. The momentum will be at the mass shell and gives the direction of the line in the configuration space. Because the parameter λ in the Hamiltonian is arbitrary we could set $\lambda = 0$ for instance and the system would not evolve at all:

$$x^\mu = x^\mu_{const} \tag{4.71}$$

$$p_\mu = (p_{const})_\mu \tag{4.72}$$

We could even choose a funny λ such that the particle oscillates or even travel back in time. This is not a problem because we saw that any time evolution in this time parameter is not physical. However we should be able to see the particle moving, this is possible when we analyze how the coordinate change with respect to the physical time that we chose to be x^0(this selection became hard in the case of General Relativity, but because we are in flat space-time this is direct). Let us analyze what are the physical observables of this theory, the first one that we would get is the momentum, but not all of them are free coordinates because of the constraint:

$$(p_\mu p^\mu + m^2) = 0 \tag{4.73}$$

We could write p_0 in terms of the other coordinates:

$$p_0^2 = \vec{p}.\vec{p} + m^2 \tag{4.74}$$

So in general we get:

$$p_0 = Q_s\sqrt{\vec{p}.\vec{p} + m^2} \tag{4.75}$$

Here $Q_s = \pm 1$ is the sign of p_0 that describes different physical states. Of course, Q_s is an observable even if the field is switched on, this is the sign of the charge of the particle. This means that in the momentum sector we have the following observables:

$$p_i = \mathcal{O}_i^1 \tag{4.76}$$

$$Q_s = sign(p_0) = \mathcal{O}_2 \tag{4.77}$$

If the field was on the observable would be the canonical momentum with the field in it. In the configuration space part, we can construct a gauge invariant function looking for the transformation of x^μ. We could try to construct an object that tells us where the particle is when the physical time is at $x^0 = t$. If an arbitrary state isn't at this time we could always make a gauge transformation that shifts the new time to this value:

$$x'^0 = t \tag{4.78}$$

Because gauge transformations don't change the physical state this is the same state. This means that $\vec{x}'$ gives us the point in space that we want. In the absence of electromagnetic field such observable can be written in the form:

$$\vec{x}_p = \vec{x} + \frac{t - x^0}{p_0} p_i = \vec{\mathcal{O}}_3 \tag{4.79}$$

We can verify that this is indeed a observable because:

$$\{\vec{x}_p, (p_\mu p^\mu + m^2)\} \approx \{\vec{x}, p_\mu p^\mu\} - \frac{\vec{p}}{p_0}\{x^0, p_\mu p^\mu\} = \tag{4.80}$$

$$= 2\vec{p} - 2\vec{p} = 0$$

This finish the complete set of observables of this theory. As was said before these functions aren't Lorentz invariant so a reduced phase space quantization would break the manifestation of the Lorentz symmetry.

4.5 Free Relativistic Particle: Dirac Covariant Quantization

The next step now is to quantize using the Dirac program, we already find the set of observables and the kinematical phase space. In this case, we don't have any ambiguity when transforming the constraint into operators in the Hilbert space. Let us work in the configuration space representation of the state:

$$\hat{x}^\mu \psi(x) = x^\mu \psi(x) \tag{4.81}$$

$$\hat{p}_\mu \psi(x) = -i\hbar \frac{\partial}{\partial x^\mu} \psi(x) \tag{4.82}$$

The constraint as a operator became:

$$\hat{C}_1 = (\hat{p}_\mu \hat{p}^\mu + m^2 \hat{I}) \tag{4.83}$$

The next step in the Dirac program is to restrict the kinematical Hilbert space given by the space where ψ lives to the physical Hilbert space where the constraint annihilate the physical states Ψ:

$$\hat{C}_1 \left|\Psi\right\rangle = 0 \tag{4.84}$$

$$(\Box - \frac{m^2}{\hbar^2})\Psi(x) = 0 \tag{4.85}$$

We know the complete set of solution of these equation so we can construct the physical Hilbert space. It is easier to construct the physical Hilbert space as eigenvalues of the momentum operator:

$$\hat{p}_i \left|q, s\right\rangle = q_i \left|q, s\right\rangle \tag{4.86}$$

$$\hat{Q}_s \left|q, s\right\rangle = s \left|q, s\right\rangle \tag{4.87}$$

In wave function representation we get:

$$\Psi_{q,s}(x) = e^{i\vec{q}.\vec{x} - is\omega x^0} \tag{4.88}$$

With:

$$\omega = \sqrt{\vec{q}.\vec{q} + \frac{m^2}{\hbar^2}} \tag{4.89}$$

Now the interesting point to make is that there is no time evolution in the nonphysical time parameter. We can't interpret the wave function as the spacial wave function that evolves in the time x^0. This causes problems like the lack of conservation of probability. The only way out that we knew until now was to go to multi-particle physics. However this is not true, we can see that the Klein Gordon equation is not the relativistic Schrodinger equation. It is not a time evolution equation for the state but a constraint. If we keep this in mind and find an adequate scalar product in the physical Hilbert space we will have no problems. We can quantize one single relativistic point particle. Now that we find the right Hilbert space we can look for the physical inner product, normally we would want that the complete set of observables that we chose to be Hermitian and this fixes the inner product up to normalization. The tri-momentum and the charge became hermitian if their eigenstates are orthogonal:

$$\left\langle q', s' \middle| q, s \right\rangle = (2\pi)^3 \delta_{c,c'} \delta^3(\vec{q} - \vec{q}')g(q) \tag{4.90}$$

If we want to fix the function $g(q)$ let us require that $\vec{x}_p$ is hermitian, here we will have a order problem because p_0 and x_0 does not commute, the most general order that we can choose is:

$$\hat{x}^i_p = \hat{x}^i - (1-\alpha)\hat{x}^0 \hat{p}_0^{-1} \hat{p}^i - \alpha \hat{p}_0^{-1} \hat{x}^0 \hat{p}^i \tag{4.91}$$

If we use the commutation relation we see that this ambiguity is really a quantum effect:

$$\hat{x}^i_p = \hat{x}^i - \hat{x}^0 \hat{p}_0^{-1} \hat{p}^i + i\hbar\alpha \hat{p}_0^{-2} \hat{p}^i \tag{4.92}$$

Because this is a observable for any value of α if we act into a physical state it should be also physical:

$$\hat{x}^i_p \left|\vec{q}, c\right\rangle = -i \frac{\partial}{\partial q^i} \left|\vec{q}, c\right\rangle + i\alpha q_i \omega^{-2} \left|\vec{q}, c\right\rangle \tag{4.93}$$

If we compute the matrix element of this we get:

$$\left\langle \vec{q}', c \right| \hat{x}^i_p \left|\vec{q}, c\right\rangle = -i(2\pi)^3 \left(\partial^i(\delta^3(q-q')g(q)) + \delta^3(q-q')\alpha q^i \omega^2 g(q)\right) \tag{4.94}$$

This operator is hermitian when:

$$\partial^i g(q) = 2\alpha q^i \omega^2 g(q) \tag{4.95}$$

If we solve this we have:

$$g(q) = \omega^{2\alpha} \tag{4.96}$$

We see that the parameter α is still free, so we can choose the most convenient one, usually this is the canonical choice $\alpha = 1/2$:

$$\left\langle \vec{q}, c \middle| \vec{q}', c' \right\rangle = (2\pi)^3 \delta_{c,c'} \delta^3(q-q')\omega \tag{4.97}$$

This concludes the treatment of the relativistic particle, we obtained the right inner product in an invariant way and the Hilbert space. One could now see that if we wanted to introduce the interaction we would need to solve the constrained equation with a background field that is an extremely hard problem:

$$\left((-i\partial^\mu + A^\mu)(-i\partial_\mu + A_\mu) + m^2\right)\Psi(x) = 0 \tag{4.98}$$

The Dirac formalism has this limitation when the constraint equation is hard to solve. There is someways to solve the constraint using physical arguments but in the end, this is where the BRST formalism start to shine. Next lecture we will work the case of the bosonic string that are a little harder but has interesting proprieties as this ambiguity that appear in the observables here appears in a more fundamental way there.

5 Lecture V

5.1 Dirac Quantization of the Bosonic String.

We will apply the Dirac quantization procedure to the Closed Bosonic String case. To do that we will need to describe the classical system and its constraints and then apply the quantization procedure. The classical action for the bosonic string is given by the worldsheet area, written in terms of the vector $X^\mu(\tau,\sigma)$. In our case lets work in the case of closed strings:

$$X^\mu(\tau,0) = X^\mu(\tau,2\pi) \tag{5.1}$$

We will choose τ as the time coordinate, then the Lagrangian is:

$$L = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma\,\sqrt{X'_\mu \dot{X}^\mu X'_\nu \dot{X}^\nu - \dot{X}_\mu \dot{X}^\mu X'_\nu X'^\nu} \tag{5.2}$$

Where prime derivative is with respect to σ. The normalization is choosed by convenience and α' is a constant with dimension of length square(the only dimensional constant in this theory that is related to the size of the string). The action written from this Lagrangian is called Nambu-Goto action and it is invariant by reparametrization in σ and τ. Now the next step is to arrive in the Hamiltonian formalism, the momenta coordinates will be:

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'}\frac{X'_\mu(X'.\dot{X}) - \dot{X}_\mu(X'.X')}{\sqrt{(X'\dot{X})^2 - (X'.X')(\dot{X}.\dot{X})}} \tag{5.3}$$

The dot is the usual Minkowski inner product. The canonical Poisson bracket is:

$$\{X_\mu(\sigma_1), \Pi_\nu(\sigma_2)\} = \eta_{\mu\nu}\delta(\sigma_1 - \sigma_2) \tag{5.4}$$

By the definition of the momenta we have the set of primary constraints:

$$C_1(\sigma) = \frac{1}{2}(4\pi^2\alpha'^2(\Pi_\mu\Pi^\mu) + (X'_\mu.X'^\mu)) \tag{5.5}$$

$$C_2(\sigma) = (\Pi_\mu X'^\mu) \tag{5.6}$$

The Hamiltonian on the constraint surface can be written as:

$$H_0 = \int \mathrm{d}\sigma\,(\dot{X}.\Pi) - L = 0 \tag{5.7}$$

As one would expect from a totally constrained system. The total Hamiltonian is:

$$H_T \approx \int \mathrm{d}sigma\,\lambda^1 C_1 + \lambda^2 C_2 \tag{5.8}$$

Lets see the gauge transformations generated by this constraints:

$$C_1(\epsilon_1) = \int \mathrm{d}\sigma\,\epsilon_1(\sigma)C_1(\sigma) \tag{5.9}$$

$$C_2(\epsilon_2) = \int \mathrm{d}\sigma\, \epsilon_2(\sigma) C_2(\sigma) \tag{5.10}$$

Acting with this on the Phase space coordinates:

$$\delta_{\epsilon_1} X^\mu = \{X^\mu(\sigma), C_1(\epsilon_1)\} = (2\pi\alpha')^2 \epsilon_1(\sigma) \Pi^\mu \tag{5.11}$$

$$\delta_{\epsilon_2} X^\mu = \{X^\mu(\sigma), C_2(\epsilon_2)\} = \epsilon_2(\sigma) X'^\mu \tag{5.12}$$

$$\delta_{\epsilon_1} \Pi_\mu = \{\Pi_\mu(\sigma), C_1(\epsilon_1)\} = \left(\epsilon_1(\sigma) X'_\mu\right)' \tag{5.13}$$

$$\delta_{\epsilon_2} \Pi_\mu = \{\Pi_\mu(\sigma), C_2(\epsilon_2)\} = (\epsilon_2 \sigma) \Pi_\mu)' \tag{5.14}$$

These constraints are the typical diffeomorphism constraints. The difference is that because we are using τ as a time coordinate the diffeomorphism constraint in the τ coordinate became dynamical, not just a shift in the coordinate. The constraint generates those transformations on the phase space that formally corresponds to the time evolution. We can see that δ_{ϵ_2} generate a shift in the phase space variables $\sigma \to \sigma + \epsilon_2$. This symmetry does not appear in the case of the relativistic particle because the spacial manifold, in that case, is just a point. This transformation is easy to find invariant functions because we just look for functions that are independent of σ. The other constraint generates a shift in the τ parameter but because we use it as the time evolution it became a dynamical evolution. Different than in the point particle case this is one constraint for each point σ in the string. This constraint usually is harder to solve because we would have to solve a functional differential equation. This first constraint replaces the time evolution generated by the Hamiltonian(which is zero in this case). These dynamical constraints are typically quadratic in the momenta, like the energy for unconstrained systems. For field theories like strings or gravity, this causes another problem. In the standard representation, the square of the momentum operator will not be well defined, as it becomes a functional differential operator. Because of that, one should think about a regularization before quantizing the string, as there will be no well-defined operator for the first constraint. The most suitable regularization is to Fourier transform the σ, which is compact and therefore we can replace integrals with sums. Doing that we can sum the divergent series that will appear and get a finite theory. Before doing the Fourier transform it is better to change the basis such that the constraints are simpler:

$$\gamma_\mu = \frac{1}{2}(2\pi\alpha' \Pi_\mu + X'_\mu) \tag{5.15}$$

$$\beta_\mu = \frac{1}{2}(2\pi\alpha' \Pi_\mu - X'_\mu) \tag{5.16}$$

In this basis the Poisson bracket are:

$$\{\gamma_\mu(\sigma_1), \gamma_\nu(\sigma_2)\} = \pi\alpha' \frac{\eta_{\mu\nu}}{2} \delta(\sigma_1 - \sigma_2) \tag{5.17}$$

$$\{\beta_\mu(\sigma_1), \beta_\nu(\sigma_2)\} = -\pi\alpha' \frac{\eta_{\mu\nu}}{2} \delta(\sigma_1 - \sigma_2) \tag{5.18}$$

$$\{\gamma_\mu(\sigma_1), \beta_\nu(\sigma_2)\} = 0 \tag{5.19}$$

They form a almost complete set of variables, they are not independent because of the total momentum of the string:

$$P_\mu = \int \mathrm{d}\sigma\, \Pi_\mu = 2\int \mathrm{d}\sigma\, \gamma_\mu = 2\int \mathrm{d}\sigma\, \beta_\mu \tag{5.20}$$

To make this a complete set we can add the center of mass coordinate for the string :

$$X_\mu^{CM} = \frac{1}{2\pi} \int \mathrm{d}\sigma\, X_\mu \tag{5.21}$$

So we describe our system using (γ, β, X^{CM}). We can write the constraints in a intelligent way using theses coordinates as:

$$C_+ = \frac{1}{2}(C_1 + 2\pi\alpha' C_2) = (\gamma.\gamma) \tag{5.22}$$

$$C_- = \frac{1}{2}(C_1 - 2\pi\alpha' C_2) = (\beta.\beta) \tag{5.23}$$

In this basis for the constraints its Poisson brackets are:

$$\{C_+(\sigma_1), C_+(\sigma_2)\} = 4\pi C_+(\sigma_2)\delta'(\sigma_1 - \sigma_2) \tag{5.24}$$

$$\{C_-(\sigma_1), C_-(\sigma_2)\} = -4\pi C_-(\sigma_2)\delta'(\sigma_1 - \sigma_2) \tag{5.25}$$

$$\{C_+(\sigma_1), C_-(\sigma_2)\} = 0 \tag{5.26}$$

We got two separated algebras of first class constraints. Because of the derivative of the Dirac delta, it is better to work in momentum space, and because of the regularization that we will need to do, then:

$$f_\mu(\sigma) = \sum_{-\infty}^{\infty} f_\mu^n e^{in\sigma} \tag{5.27}$$

$$f_\mu^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_\mu(\sigma) e^{-in\sigma} \tag{5.28}$$

Because the function is real we get that:

$$(f_\mu^n)^* = f_\mu^{-n} \tag{5.29}$$

With this convention the Dirac delta became:

$$\delta(\sigma_1 - \sigma_2) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{in(\sigma_1 - \sigma_2)} \tag{5.30}$$

In the Fourier space the canonical Poisson brackets are:

$$\{X^n_\mu, \Pi_\nu(m)\} = \frac{1}{2\pi} \eta_{\mu\nu} \delta_{n+m} \tag{5.31}$$

In the choosed basis:

$$\{\gamma^n_\mu, \gamma^m_\nu\} = \frac{i\alpha' n}{2} \eta_{\mu\nu} \delta_{n+m} \tag{5.32}$$

$$\{\beta^n_\mu, \beta^m_\nu\} = -\frac{i\alpha' n}{2} \eta_{\mu\nu} \delta_{n+m} \tag{5.33}$$

$$\{\gamma^n_\mu, \beta^m_\nu\} = 0 \tag{5.34}$$

We can write the constraints in the Fourier space as well:

$$C^n_+ = \sum_m \gamma^\mu_m \gamma^{n-m}_\mu \tag{5.35}$$

$$C^n_- = \sum_m \beta^\mu_m \beta^{n-m}_\mu \tag{5.36}$$

The constraints are now in a more familiar form, known as the Witt algebra:

$$\{C^n_+, C^m_+\} = i\alpha'(n-m) C^{n+m}_+ \tag{5.37}$$

$$\{C^n_-, C^m_-\} = -i\alpha'(n-m) C^{n+m}_- \tag{5.38}$$

$$\{C^n_+, C^m_-\} = 0 \tag{5.39}$$

We are now ready to quantize the theory. Now we construct a wave functional in the canonical basis such that:

$$\hat{X}^n_\mu \Psi = X^n_\mu \Psi \tag{5.40}$$

$$\hat{\Pi}^n_\mu \Psi = -2\pi i\hbar \frac{\partial \Psi}{\partial X^{-n}_\mu} \tag{5.41}$$

The basis that we choosed is easily converted for quantum operators:

$$\hat{\gamma}^n_\mu = \frac{1}{2}(2\pi\alpha' \hat{\Pi}^n_\mu + in\hat{X}^n_\mu) \tag{5.42}$$

$$\hat{\beta}^n_\mu = \frac{1}{2}(2\pi\alpha'\hat{\Pi}^n_\mu - in\hat{X}^n_\mu) \tag{5.43}$$

The next step that we need to do is to transform the constraint into a quantum operator. The classical function is defined in (5.35) and (5.36). A interesting thing is happening, for $n \neq 0$ the operators that appear are commuting so there is no order ambiguity in the quantization of them. For $n = 0$ each term on the sum does not commute and the order ambiguity that we discussed before emerges here. Lets work out the $n = 0$ case, we can write both the constraints as (choosing a symmetric ordering):

$$\hat{C}_+(0) = \frac{1}{16\pi^2}(\hat{P}.\hat{P}) + \sum_{m>0}\left(\hat{\gamma}^m_\mu \hat{\gamma}^\mu_{-m} + \hat{\gamma}^{-m}_\mu \hat{\gamma}^\mu_m\right) \tag{5.44}$$

$$\hat{C}_-(0) = \frac{1}{16\pi^2}(\hat{P}.\hat{P}) + \sum_{m>0}\left(\hat{\beta}^m_\mu \hat{\beta}^\mu_{-m} + \hat{\beta}^{-m}_\mu \hat{\beta}^\mu_m\right) \tag{5.45}$$

Where $\hat{P}$ is the operator of the momentum of the center of mass. We could write using complex conjugation as well:

$$(\hat{\gamma}^n_\mu)^\dagger = (\hat{\gamma}^{-n}_\mu) \tag{5.46}$$

It is clear that in this form there will never be a solution for the constraints. This sum will no converge when acting on the same state. We have to reorder the operators such that both sums can converge. We can see that in the $n > 0$ case γ^n_μ and β^{-n}_μ should be the creation operators and the hermitian conjugate the annihilation one, only from the algebra. Almost all terms must be such that the annihilation operator acts first, as otherwise there is no chance for the vacuum to be a physical state. Now we can re-order many times as we want and each time we pick up a constant. The final result is:

$$\hat{C}_+(0) = \frac{1}{16\pi^2}(\hat{P}.\hat{P}) + 2\sum_{m>0}\left(\hat{\gamma}^m_\mu \hat{\gamma}^\mu_{-m}\right) + \hbar\Omega \tag{5.47}$$

$$\hat{C}_-(0) = \frac{1}{16\pi^2}(\hat{P}.\hat{P}) + 2\sum_{m>0}\left(\hat{\beta}^m_\mu \hat{\beta}^\mu_{-m}\right) + \hbar\Omega \tag{5.48}$$

In a first look the constants could be different but as we will see ahead there will be a restriction to that. We could arrive at the exact value for N, if coming from a symmetric ordering this is even easier. The only thing that will be necessary to do is to sum a divergent series that will appear. The focus here will not be the actual value of N but the algebra of the constraint. This additional term generates an anomaly in the classical algebra and we get the Virasoro algebra. Doing the computation directly is painful, we will use the Jacobi identity and the behavior of creation and annihilation operator to find the result in a cleaver way. In the case that n+m is different than zero we should expect that the algebra remains the same so without losing any generality we can write:

$$[\hat{C}_+^n, \hat{C}_+^m] = -\alpha'(n-m)\hat{C}_+^{n+m} + A(m)\delta_{n+m} \tag{5.49}$$

The term $A(m)$ is the anomalous term and because the proprieties of the bracket we have $A(m) = -A(-m)$ and its a complex number that only depends on m. This will be similar to the C_- algebra. We can find its dependence on m by using the Jacobi Identity:

$$[\hat{C}_+^k, [\hat{C}_+^n, \hat{C}_+^m]] + [\hat{C}_+^n, [\hat{C}_+^m, \hat{C}_+^k]] + [\hat{C}_+^m, [\hat{C}_+^k, \hat{C}_+^n]] = 0 \tag{5.50}$$

This gives us for $n+m+k=0$:

$$-\alpha'[(n-m)A(k) + (m-k)A(n) + (k-n)A(m)] = 0 \tag{5.51}$$

If we choose k=1 we can find a recurrence relation:

$$A(n+1) = \frac{(n+2)A(n) - (2n+1)A(1)}{(n-1)} \tag{5.52}$$

The most general solution for this relation is:

$$A(m) = c_1 m + c_2 m^3 \tag{5.53}$$

Now if we want to determine this constants we can compute the expectation value in the vacuum of:

$$\langle 0|[\hat{C}_+^n, \hat{C}_+^{-n}]|0\rangle \tag{5.54}$$

Because the choice of creation and annihilation operators we have that:

$$\gamma_\mu^n |0\rangle = 0 \; n < 0 \tag{5.55}$$

$$\beta_\mu^n |0\rangle = 0 \; n > 0 \tag{5.56}$$

Then we can compute for n=1 and this give a relation between c_1 and c_2

$$\langle 0|[\hat{C}_+^1, \hat{C}_+^{-1}]|0\rangle = 0 \tag{5.57}$$

$$c_1 = -c_2 \tag{5.58}$$

Because both $\hat{C}_+^1$,$\hat{C}_+^{-1}$ annihilate the vacuum, all the terms give zero. For n=2 we have:

$$\langle 0|[\hat{C}_+^2, \hat{C}_+^{-2}]|0\rangle = -\langle 0|C_+^{-2}C_+^2|0\rangle = -\langle 0|((\hat{\gamma}_{-1}.\hat{\gamma}_{-1})(\hat{\gamma}_1.\hat{\gamma}_1|0\rangle \tag{5.59}$$

These are the only non zero terms, now we get the annihilation operators to the right side using the commutation relations:

$$\langle 0|[\hat{C}_+^2, \hat{C}_+^{-2}]|0\rangle = -\frac{\hbar^2\alpha'^2 D}{2} \tag{5.60}$$

Where D is the space-time dimension. Now we can equal both sides to find $A(m)$:

$$A(m) = -\hbar^2\alpha'^2\left(\frac{D}{12}(m^3-m) - 2m\Omega\right) \tag{5.61}$$

The additional term comes by compensation the C_+^0 on the right side of the commutator. With a similar argument we get the other algebra as well:

$$[\hat{C}_+^n, \hat{C}_+^m] = -\hbar\alpha'(n-m)C_+^{n+m} - \hbar^2\alpha'^2\delta_{n+m}\hbar^2\alpha'^2\left(\frac{D}{12}(m^3-m) - 2m\Omega\right) \tag{5.62}$$

$$[\hat{C}_-^n, \hat{C}_-^m] = \hbar\alpha'(n-m)C_-^{n+m} + \hbar^2\alpha'^2\delta_{n+m}\hbar^2\alpha'^2\left(\frac{D}{12}(m^3-m) - 2m\Omega\right) \tag{5.63}$$

As is clear the algebra does not close anymore and even the freedom to choose Ω can't make it close. This means that the procedure to impose:

$$\hat{C}\,|\Psi\rangle = 0 \tag{5.64}$$

Does not select physical states. The system is no longer first class and does not make any physical sense. In this case we need to use a different arrange of constraints. It is possible to construct two complex conjugate set of constraints that have closed algebra, this is fixed and we don't have any freedom now. We want that a physical state have the propriety:

$$\hat{\gamma}_\mu^n\,|0\rangle = 0 \text{ for } n < 0 \tag{5.65}$$

$$\hat{\beta}_\mu^n\,|0\rangle = 0 \text{ for } n > 0 \tag{5.66}$$

Such that we can call this state the physical vacuum. It is direct to see that if we choose the physical condition to be:

$$\hat{C}_+^n\,|\Psi\rangle = 0 \text{ for } n \leqslant 0 \tag{5.67}$$

$$\hat{C}_-^n\,|\Psi\rangle = 0 \text{ for } n \geqslant 0 \tag{5.68}$$

And the Hermitian conjugate, except for $n = 0$ where the constraints are real. These set of constraints close even if we include $n = 0$ in both subsets:

$$[\hat{C}_+^n, \hat{C}_+^m]\,|\psi\rangle = 0 \text{ for } n \leqslant 0\ ,\ m \leqslant 0 \tag{5.69}$$

In the case where $n + m < 0$ the anomaly term never acts and when its equal the anomaly term is zero. Now the Fock space can be constructed. We can solve the wave function for the ground state for $n \neq 0$:

$$(2\pi\alpha'\hat{\Pi}_\mu^n + in\hat{X}_\mu^n)\,|0\rangle = 0 \text{ for } n < 0 \tag{5.70}$$

$$(2\pi\alpha'\hat{\Pi}^n_\mu - in\hat{X}^n_\mu)\,|0\rangle = 0 \text{for } n > 0 \tag{5.71}$$

Then we just need to solve:

$$\hat{\Pi}^n_\mu\,|0\rangle = \frac{i|n|}{2\pi\alpha'}\hat{X}^n_\mu\,|0\rangle \tag{5.72}$$

Using how the momentum act in the state we can see that the solution will be:

$$\Psi = \Psi(X^{CM})e^{-\frac{1}{2\hbar\alpha'}\sum_m X^\mu_n X^{-n}_\mu} \tag{5.73}$$

Where $\Psi(X^{CM})$ only depend on the center of mass of the string. It is easy to see that asking to the constraint to be satisfy we get:

$$(\hat{P}^2 + 16\pi^2\hbar\Omega)\Psi(X^{CM}) = 0 \tag{5.74}$$

So the ground state of the string behaves like a particle with a mass proportional to the ordering constant that is not fixed yet:

$$m^2 = 16\pi^2\hbar\Omega \tag{5.75}$$

We will not compute N but if this is done we would find that the theory is only consistent if $D \leqslant 26$ and $\Omega \leqslant -2\alpha'$. If we introduce interaction this restricts, even more, the possible values and until now it is only know how to introduce interactions in $D = 26$ $\Omega = -2\alpha'$, this is called the critical dimension of the string and the ground state has a tachyon. In this theory, there is an additional problem that the creation and annihilation operators are not gauge invariant only the small subset around n=0. This means that acting with them will take a physical state into a nonphysical one. We would need to work the equivalence classes again because the gauge freedom is not completely fixed. This is just a problem of the choice of creation and annihilation operators, if we used one that does not respect Lorentz than this would be possible. Nevertheless, the space that we constructed is consistent even with the appearance of the anomaly term in the algebra. This computation was more to show what happens in theories with anomalies and how is sometimes possible to deal with that. Of course, the problem that we have in the end could be avoided if we used BRST quantization because, in the end, we would gauge fix everything and kill all de freedom consistently. The interesting point that this procedure brings is that everything is gauge invariant until the end. Here we only quantized the closed bosonic string but the open case is trivially archived using the same procedure only with half of the constraints. This is just the beginning of the beginning of strings, the study of interactions and superstrings that lies ahead need more powerful tools that can be studied once the Dirac Program is well understood. Next lecture we will work the case of the Free Quantum Electromagnetic field and conclude the course.

6 Lecture VI.

6.1 Dirac Quantization of Electromagnetic Field.

We saw how to treat constraints of first and second class classically and quantically. We work out some simple examples before attacking the case of systems that are invariant by diffeomorphism in one and two dimensions as a prototype to quantize gravity or even dive deeper in string theory. Now lets finally apply this knowledge to a more "high energy" problem, let's quantize the electromagnetic field. To do that we will use again the Dirac program of quantization to arrive in a quantum theory that is gauge invariant. After this last example, we will discuss a little about the Dirac Program in general and see the road ahead of us. Normally, in a usual course of quantum field theory, the quantization of the electromagnetic field is done in rush and in a nonsatisfactory manner in the canonical formalism. Only when the path integral is introduced that the quantization is done in a more rigorous and consistent manner and generalized to Yang-Mills Theories in general. What we will do here is to show the power of the Dirac program to construct the Fock space of that theory without fixing the gauge. The first step to do that is starting from the free action of the Maxwell Field:

$$S = \int \mathrm{d}^4x \, (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}) \tag{6.1}$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We know that this action is invariant under the gauge transformation:

$$\delta A_\mu = \partial_\mu \Lambda \tag{6.2}$$

This transformation uses Λ and its time derivative this means that the gauge acts twice killing two degrees of freedom. Lets see how this symmetry manifest itself in the Hamiltonian formalism. First we split the spacial and temporal parts of the Lagrangian:

$$\mathcal{L}(\vec{A}, \dot{\vec{A}}) = \int \mathrm{d}^3x \, \frac{1}{4} \, (2F_{0i}F_{0i} - F_{ij}F_{ij}) \tag{6.3}$$

To follow the Dirac formalism we need that $t = x^0$ is distinguished. Now we need to go to Hamiltonian description, the canonical momentum are:

$$\Pi^i(x) = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = F^{0i} \tag{6.4}$$

$$\Pi^0 = 0 \tag{6.5}$$

Immediately have the first class constraint

$$C_1 = \Pi^0 \tag{6.6}$$

Because the field nature this constraint is actually a infinite set of constraint one for each space point. This canonical variables have the usual same time Poisson bracket:

$$\{A_\mu(x), \Pi_\nu(y)\} \approx \eta_{\mu\nu} \delta^3(x - y) \tag{6.7}$$

The Hamiltonian is easily written as:

$$H_T \approx \int \mathrm{d}^3x \left(\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}F_{ij}F_{ij} + \Pi_i\partial_i A_0 + \lambda_1\Pi^0 \right) \tag{6.8}$$

With the Lagrange multiplier λ_1 being dependent of space as well. Now we need to check if the time evolution preserves this constraint:

$$\{C_1(x), H_T\} \approx \int \mathrm{d}^3z \, \{\Pi^0(x), -\partial_i\Pi_i A_0\} = -\partial_i\Pi_i(x) \tag{6.9}$$

Then for consistency we need to put this secondary constraint in our Hamiltonian:

$$C_2(x) = -\partial_i E_i \tag{6.10}$$

This is the usual Gauss law that here appear as a constraint. So our Extended Hamiltonian is:

$$H_E \approx \int \mathrm{d}^3x \left(\frac{1}{2}\Pi_i\Pi_i + \frac{1}{4}F_{ij}F_{ij} + \Pi_i\partial_i A_0 + \lambda_1\Pi^0 - \lambda_2\partial_i\Pi_i \right) \tag{6.11}$$

Now it is clear that both constraints are first class and there are no more constraints:

$$\{C_1(x), C_2(y)\} \approx 0 = 0 \tag{6.12}$$

$$\{C_2(x), H\} = -\partial_i\partial_j F^{ij} \approx 0 = 0 \tag{6.13}$$

Where we used the antisymmetry of the F. It is clear that both Lagrange multipliers are arbitrary, lets see the gauge transformation generated by this first class constraint, if we write:

$$C(\epsilon_1, \epsilon_2) = \int \mathrm{d}^3x \, \epsilon_1(x)C_1(x) + \epsilon_2(x)C_2(x) \tag{6.14}$$

Then:

$$\delta A_0 = \{A_0, C\} = \epsilon_1 \tag{6.15}$$

$$\delta A_i = \{A_i, C\} = \partial_i\epsilon_2 \tag{6.16}$$

This symmetry is not in the expected form that we saw in the Lagrangian formalism, lets see how the Lagrange multipliers transform and then recover the form of the Lagrangian transformation fixing the secondary constraint. The extended action is of the form:

$$S_E \approx \int \mathrm{d}^4x \, [\Pi_i\dot{A}_i - \Pi_0 A_0 - H_C - \lambda_1\Pi^0 + \lambda_2\partial_i\Pi_i] \tag{6.17}$$

Under the transformation it is easy to see that:

$$\delta\lambda_1 = \dot{\epsilon_1} \tag{6.18}$$

$$\delta\lambda_2 = \dot{\epsilon_2} - \epsilon_1 \tag{6.19}$$

Then if we want to recover the form of the transformation from the Lagrangian formalism we fix the gauge in such a way that:

$$\lambda_2 = 0 \tag{6.20}$$

$$\delta\lambda_2 = 0 \tag{6.21}$$

This says that $\dot{\epsilon_2} = \epsilon_1$ and the symmetry is exactly what we had before. Now let's stay in the Hamiltonian formalism and focus in the quantization of this theory. We just need to find classical gauge invariant functions that we will use to describe the system. In this case, it is easy to see that the momenta are invariant and the magnetic field will be as well:

$$\{E_i, C\} = 0 \tag{6.22}$$

$$\{B_i, C\} = 0 \tag{6.23}$$

In which $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ and $E_i = \Pi_i$. It is worth remember that the extended Hamiltonian generate equations of motion that are equal only in the constraint surface. Because of that, F_{0i} is not the electric field in this formalism because the secondary constraint added. In the end F_{0i} will be the electric field and will match with the momentum. The energy is the Hamiltonian on the constraint surface and we recover the usual one:

$$H = \int \mathrm{d}^3x \, \frac{1}{2}\left(E_iE_i + B_iB_i\right) \tag{6.24}$$

Now that the classical theory is solved lets quantize the theory. The first problem of a field theory is that the wave function now became a wave functional. We define the basis were the $\hat{A}_\mu$ acts as multiplicative operator:

$$\hat{A}_\mu \Psi = A_\mu \Psi \tag{6.25}$$

$$\hat{E}_\mu \Psi = -i\hbar\eta_{\mu\nu}\frac{\delta\Psi}{\delta A_\nu} \tag{6.26}$$

The constraints became quantum operators in the Fock space, there is no problem of ordering because of the linear nature:

$$\hat{C}_1 \Psi = 0 \tag{6.27}$$

$$\hat{C}_2 \Psi = 0 \tag{6.28}$$

In the representation choosed:

$$\frac{\delta\Psi}{\delta A_0(x)} = 0 \tag{6.29}$$

$$\partial_i \frac{\delta\Psi}{\delta A_i(x)} = 0 \tag{6.30}$$

This constraints define the physical Fock space, the first one says that a physical state cannot depend on A_0. The second condition says that this state need to be gauge invariant:

$$\delta\Psi = \int \mathrm{d}^3z\, \delta A_i \frac{\delta\Psi}{\delta A_i} = -\int \mathrm{d}^3z\, \epsilon(z)\partial_i \frac{\delta\Psi}{\delta A_i} = 0 \tag{6.31}$$

Now what we need to do is to construct such Fock space. A useful procedure is to find one state that satisfy the constraints, then act with observables in this state. When act using observables this will generate another physical state. As the physical state space is required to be an irreducible representation of the observable algebra, we can generate every physical state in this way. In the end we will construct the Fock space in some representation. This is not a guarantee that will work as sometimes the Fock space constructed will not be normalizable for instance. We can try to construct creation and annihilation operators that are gauge invariant using the electric and magnetic field. By their definition they follow the algebra:

$$[\hat{E}_i(x), \hat{B}_j(y)] = -i\hbar\epsilon_{ijk}\partial_k\delta^3(\vec{x}-\vec{y}) \tag{6.32}$$

Or in the Fourier space:

$$[\hat{E}_i(k), \hat{B}_j(q)] = -\hbar(2\pi)^3\epsilon_{ijk}k_k\delta^3(k+q) \tag{6.33}$$

Where $\hat{E}_i(k)^\dagger = \hat{E}_i(-k)$. A first candidate to operator of creation and annihilation appear if we write the Hamiltonian choosing a ordering in the way:

$$H = \frac{1}{2}\int \mathrm{d}^3x\, (\hat{E}_i + i\hat{B}_i)(\hat{E}_i - i\hat{B}_i) \tag{6.34}$$

There is a ambiguity in this definition but we always can drop out any constant here because we only measure differences in energies(fuck you gravity!). In this way we can try to use:

$$\hat{a}_i = (\hat{E}_i - i\hat{B}_i) \tag{6.35}$$

$$\hat{a}_i^\dagger = (\hat{E}_i + i\hat{B}_i) \tag{6.36}$$

If we do this choice and use hermitian fields we define the vacuum as:

$$\hat{a}_i \left|0\right\rangle = 0 \tag{6.37}$$

$$\langle 0|0\rangle = 1 \tag{6.38}$$

Now we can construct all the Fock states acting with the creation operator in the vacuum. These states will be physical by construction but lets see if they have positive definite norm. A state created by the creation operator is of the form:

$$\left|\psi\right\rangle = \int \mathrm{d}^3x\, \psi_i(x) a_i^\dagger(x) \left|0\right\rangle \tag{6.39}$$

We can already expect trouble because the commutator of the creation and annihilator operators are not positive definite:

$$[a_i(x), a_j^\dagger(y)] = 2\hbar\epsilon_{ijk}\partial_k\delta^3(\vec{x} - vecy) \tag{6.40}$$

The norm of the state is:

$$\langle\psi|\psi\rangle = 2\hbar\epsilon_{ijk}\int \mathrm{d}^3x\,\psi_i^*\partial_k\psi_j \tag{6.41}$$

This product is not positive definite, if we choose a appropriate function ψ_i we can make this norm be an arbitrary negative number. This means that this is not a good inner product and we can't have a normed state and the conjugation proprieties at the same time with these operators. We should look for another vacuum that fulfills the constraint and create another set of creation and annihilation operators. To do that we will go to the Fourier space and use:

$$\hat{a}_i(k) = \hat{E}_i(k) + \epsilon_{ijk}\frac{k_j}{|k|}B_k(k) \tag{6.42}$$

$$\hat{a}_i^\dagger(k) = \hat{E}_i^*(k) + \epsilon_{ijk}\frac{k_j}{|k|}B_k^*(k) \tag{6.43}$$

This form can be justified because it generate the Hamiltonian given an ordering but more important it has a positive semi definite commutation relation:

$$[\hat{a}_i(k), \hat{a}_j^\dagger(q)] = 2\hbar(2\pi)^3|k|P_{ij}(k)\delta^3(\vec{k} - \vec{q}) \tag{6.44}$$

Where $P_{ij} = \frac{\delta_{ij}k^2 - k_ik_j}{k^2}$ is the operator of transverse projection. With these operators we can define the vacuum:

$$\hat{a}_i\,|0\rangle = 0 \tag{6.45}$$

In this case we will generate all the Fock space acting with the creation operator in that vacuum. Lets verify if these states will have positive norm. We define the state create by one operator as:

$$|\psi\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\psi_i(k)a_i^\dagger(k)\,|0\rangle \tag{6.46}$$

The norm of that state is:

$$\langle\psi|\psi\rangle = 2\hbar\int \frac{\mathrm{d}^3k}{(2\pi)^3}(k^2\delta_{ij} - k_ik_j)\psi_i^*\psi_j \tag{6.47}$$

Now we have a positive semi-definite norm. We still have to show that the zero norm state is the trivial one. The state will have zero norm when $\psi_i = k_i\theta(k)$. In this case we can see that inside the physical space they will be indeed the trivial state:

$$|\psi\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\theta(k)k_i\hat{a}_i^\dagger(k)\,|0\rangle = 0 \tag{6.48}$$

Where we used the Fourier transform of the Gauss law constraint. Then inside the physical Fock space, this norm is positive definite. Then our work is done we found all the physical Fock states in a gauge invariant approach. It is possible to see that these functions that define the n particle states have some ambiguities coming from the gauge invariance of the vacuum. We could fix this redundancy but this is not mandatory because any matrix element in this theory will be gauge invariant. The operators guarantee us that a physical state does not evolve to a nonphysical state. Then we have the free theory quantized in a gauge invariant manner and we could compute expected values on it. Of course, the addition of interaction would be problematic as the Gauss law constraint would be harder to solve but in principle, we could do the same thing and get a gauge invariant theory in the end.

6.2 Conclusion

In this course, we learned how to quantize theories with constraints. We work out in more detail the first class constraints because is the most relevant case for Modern Physics. Using this knowledge we quantized some smaller examples and most importantly the free particle, bosonic string, and the electromagnetic field. Each case had its peculiarities and difficulties inherent from the formalism. We focused during the lectures in the Dirac quantization program, that deal with the system in a gauge invariant manner until the end. This by itself is a good advantage for the formalism but there are some flaws in it. The biggest problem of that formalism is when the equation for the constraint is hard. Usually, we can't solve a differential equation so there is a large chance that we crash in a wall doing this formalism. Because of this problem that more powerful techniques were developed. The BRST method is born when we give up the gauge invariance but in an intelligent manner, we enlarge the phase space in such a way that we can use the techniques of the Dirac procedure in that enlarged space to separate physical states. There are even powerful methods like BV quantization that bring the power of BRST to the Lagrangian formalism where most of the symmetries are manifest. With the knowledge acquired here, it is possible to start the road to BRST in a more clear and direct way. During the treatment of the constraint in this lecture we assumed that they are irreducible, something that is not always the case. These details can appear in complicated theories and it should be dealt with great care. I just want to emphasize that it is very important to understand the difference of the constraint in the Lagrangian and Hamiltonian formalism, something that even modern physicists get confused from time to time. The study of constraint system in the way that is presented here is still an open field with some interesting open questions like what axioms we need to guarantee the Dirac conjecture or how to deal in an invariant manner with mixed constraints. To conclude, this was a brief introduction in this topic that will appear somewhere in the path of many physicists so the basis constructed here can be used to study many modern problems.

A Geometric Formulation of Classical Mechanics

Here classical mechanics will be defined in a geometrical way that will be useful to visualize some important features. In this spirit the configuration space of a n−degree of freedom as a n−dimensional manifold $\mathcal{M}$ with local coordinates $(q^1(t), ..., q^n(t))$. Here the parameter t is the real time so a trajectory is then the time evolution of the system. To describe the system we need information of about the velocity, to have that we can construct the velocity phase space as a tangent bundle over $\mathcal{M}$. It has local coordinates $(q^1(t), \ldots, q^n(t), \dot{q}^1(t), \ldots, \dot{q}^n(t))$ and is a $2n$−dimensional manifold. The dynamics in this picture is specified by a Lagrangian $\mathcal{L}$:

$$\mathcal{L} \in Fun\,(T\mathcal{M}) \tag{A.1}$$

This Lagrangian determines the dynamics of the vector field X in the manifold.

$$X = \frac{d}{dt} \tag{A.2}$$

With the vector field determined we can find its integral curves $\gamma(t)$ and by the means of canonical projection[7] get the motion on the manifold:

$$\delta(t) = \Pi\,(\gamma(t)) \tag{A.3}$$

The vector field in the velocity phase space on local coordinates is[8]:

$$X = \frac{d}{dt} = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} \tag{A.4}$$

The vector field is determinate by the Lagrangian function $\mathcal{L}$, to find the equation that describe it is useful to construct the Lagrangian 1−form θ:

$$\theta = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} dq^i \tag{A.5}$$

And with that construct a symplectic 2−form in $T\mathcal{M}$, $\omega = d\theta$[9] and a energy function:

$$E = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L} \tag{A.6}$$

The equation that fixes the vector field X is the usual Euler−Lagrange equation:

$$\mathscr{L}_X\,(\theta) = d\mathcal{L} \tag{A.7}$$

In this equation $\mathscr{L}_X$ is the lie derivative with respect of the vector field X, to see that this is indeed the Euler-Lagrange equations it is easy to compute:

$$\mathscr{L}_X \theta = \mathscr{L}_X \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) dq^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \mathscr{L}_X \left(dq^i \right) \tag{A.8}$$

[7] Canonical projection is just the act of dropping the velocity coordinates.

[8] Here we identify $\frac{dq^i}{dt} = \dot{q}^i$.

[9] Here q and $\dot{q}$ are treated as independent.

Now using that:

$$\mathscr{L}_X \left(dq^i\right) = d\mathscr{L}_X \left(q^i\right) = d\dot{q}^i \tag{A.9}$$

$$\mathscr{L}_X \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \tag{A.10}$$

We an write:

$$\mathscr{L}_X \theta = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} dq^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} d\dot{q}^i \tag{A.11}$$

If the Euler−Lagrange equation is satisfy then:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial q^i} \tag{A.12}$$

Using this we prove that the first equation is only solved when Euler−Lagrange equations are valid:

$$\mathscr{L}_X \theta = \frac{\partial \mathcal{L}}{\partial q^i} dq^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} d\dot{q}^i = d\mathcal{L} \tag{A.13}$$

This formulation is useful in describing relativistic systems and others things but has a less clear geometrical interpretation and when the variables aren't free it has limitations. Because of this, to move to the analysis of constraints in classical mechanics lets introduce the Hamiltonian description. The main point in moving from Lagrangian to Hamiltonian prescription is to describe the system using the cotangent bundle $T^*\mathcal{M}$, where the local coordinates are $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ and the canonical 1−form momentum is:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \tag{A.14}$$

This space is called phase space (Sometimes called kinematics phase space because it can have more degrees of freedom than the actual physical system). To describe the dynamics in this picture we need a function on phase space that dictates the dynamics. With the knowledge of the Lagrangian formalism, it is possible to construct such a function on phase space as:

$$H(q, p, t) = p_i \dot{q}^i - \mathcal{L} \tag{A.15}$$

An important point is that to this description make any sense you should be able to write $\dot{q}^i$ as a function of the momentum, basically inverting relation (A.14). This is not always the case, as we will see when you have constraints this procedure fails and we need to treat the system with more care. The great thing about the Hamiltonian description is the trade of n second order equation for $2n$ first order Hamiltonian equation. The process of going from Lagrangian to Hamiltonian mechanics is called a Legendre transformation:

$$Fun\,(T\mathcal{M}) \leftrightarrow Fun\,(T^*\mathcal{M}) \tag{A.16}$$

In the phase space if the Hamiltonian does not depend on time there is a natural closed symplectic nondegenerate $2-$form ω[10]. This means that the phase space is actually a symplectic manifold and you can canonically relate vectors and co$-$vectors on it. Having ω and being nondegenerate we can define its inverse Ω, in matrix form:

$$\Omega^{ab}w_{bc} = \delta^a_c \tag{A.17}$$

A important object that lives in a sympletic manifold is a Hamiltonian vector field, defined as[11]:

$$X_f = \Omega \lrcorner df \tag{A.18}$$

In components:

$$X_f^a = \Omega^{ab}\partial_b f \tag{A.19}$$

Or we could invert this and write as:

$$X_f \lrcorner \omega = -df \tag{A.20}$$

This vector field will generate its integral curves and the integral curves will generate the flow in the phase space. With the inverse of ω we can define canonically a Poisson bracket:

$$\{f, g\} = df \lrcorner \Omega \lrcorner dg = -w(X_f, X_g) \tag{A.21}$$

A important proprieties is that:

$$X_{\{f,g\}} = [X_f, X_g] \tag{A.22}$$

The last thing that we need to do Hamiltonian mechanics is the Darboux coordinate system. It is proven that you can always can find a basis in phase space such that the symplectic form is in the canonical form[12]

$$\omega = dp_i \wedge dq^i \tag{A.23}$$

Grouping all this elements, the Hamiltonian function defined in (A.15) defines a Hamiltonian vector field:

$$X_H = \frac{d}{dt} = \Omega \lrcorner dH = \{, H\} \tag{A.24}$$

[10]If the Hamiltonian depends on the time the right description is in terms of contact geometry that has a degenerate $2-$form instead of the non-degenerate of this case. This makes impossible to have an inverse and the manifold does not have a natural way to relate vectors and co$-$vectors.

[11]$\lrcorner$ is the inner derivative, sometimes written as i_X. $i_X\omega = X \lrcorner \omega$.

[12]Transformations that preserve this coordinate system are called canonical transformations and plays an important role in classical mechanics.

This field generate integral curves:

$$\gamma(t) = \left(q^i(t), p_i(t)\right) \tag{A.25}$$

The Hamiltonian vector field X_H is defined as tangent to this curve, that gives us:

$$X_H = \frac{d}{dt} = \frac{\partial q^i}{\partial t}\frac{d}{dq^i} + \frac{\partial p_i}{\partial t}\frac{d}{dp_i} \tag{A.26}$$

But at the same time from the definition (A.24):

$$X_H = \{,H\} = \frac{\partial H}{\partial p_i}\frac{d}{dq^i} - \frac{\partial H}{\partial q^i}\frac{d}{dp_i} \tag{A.27}$$

Matching both relation we can see that the Hamiltonian equations of motion are satisfy:

$$\frac{\partial H}{\partial p_i} = \dot{q}^i \tag{A.28}$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{A.29}$$

Finally solving H we can find the integral curve $\gamma(t)$. To get only the position solution you just project down to the manifold to get the trajectory $\delta(t)$ using canonical projection:

$$\delta(t) = \Pi\left(\gamma(t)\right) \tag{A.30}$$

A direct consequence of this formalism is that the time evolution generated by H preserves volume in phase space. The last element of the Hamiltonian dynamics that we need is the Noether theorem that relates symmetries with conserved quantities.

A.1 Noether Theorem (Hamiltonian version)

Let $\Upsilon \in T\left(T^*\mathcal{M}\right)$ a vector field on phase space such that preserves ω and H:

$$\mathscr{L}_\Upsilon \omega = 0 \tag{A.31}$$

$$\mathscr{L}_\Upsilon H = 0 \tag{A.32}$$

Then exists a function $I(q,p)$ such that:

$$\mathscr{L}_{X_H} I = 0 = \frac{dI}{dt} \tag{A.33}$$

To prove this lets use cartan's lemma:

$$\mathscr{L}_V \omega = V \lrcorner dw + d\left(V \lrcorner \omega\right) \tag{A.34}$$

With this lemma the first relation (A.31) became:

$$\mathscr{L}_\Upsilon \omega = 0 = \Upsilon \lrcorner d\omega + d(\Upsilon \lrcorner \omega) \tag{A.35}$$

The first term is automatically zero because ω is closed, the second term can be written locally as:

$$\Upsilon \lrcorner \omega = -dI \tag{A.36}$$

This relation is the definition of a Hamiltonian vector field of the function I, using this on (A.32) we get:

$$\mathscr{L}_{\Upsilon} H = 0 = \mathscr{L}_{\Upsilon_I} H = 0 \tag{A.37}$$

With this information we automatically get:

$$\mathscr{L}_{\Upsilon} H = 0 = \{H, I\} = -\{I, H\} \tag{A.38}$$

We arrived at a powerful relation, this means that I is a constant of motion but at the same time, H is invariant under the flow generated by I. This version of Noether theorem talks only about the symmetry of the phase space, depending on the nature of Υ this symmetry can be realized in the configuration space or not.