Charge quantisation without magnetic poles

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November 10, 2017

Main reference

- Charge quantisation without magnetic poles: a topological approach to electromagnetism.

  Journal of Geometry and Physics (2016).

Organisation

- Present the natural phenomenon of electric charge quantisation.

- Pose the problem of electric charge quantisation
  (and hint a solution).

- Motivate the new formalism.

- Derive the consequences of the new formalism.

Fact from nature

- Fletcher and **Millikan** (1909–1913):

$$q_e = ke \ .$$

Electric charge quantisation problem

- Does there exist an **electromagnetic theory** able to **explain** the natural phenomenon of the electric **charge quantisation without** extra **unobserved physical hypothesis**?

- Idea: associate **curvature** with $\star F \;\Rightarrow$ **no** magnetic poles.

- Challenge: $\star F$ does **not** satisfy the Bianchi identity,

  better saying $\int_{\partial\mathcal{V}} \star F$ is **not zero**, unless $\rho_e = 0$ and $\boldsymbol{J}_e = 0$.

Contemporary Maxwell's theory: magnetic poles and Faraday

- **Faraday tensor**, F:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} .$$

- **No** magnetic poles and **Faraday's law of induction**,

$$\int_\Sigma F = 0 \ \Rightarrow \ \partial_{[\lambda} F_{\mu\nu]} = 0 \ .$$

Contemporary Maxwell's theory: Gauss and Ampère

- **Hodge dual** $\star F$ of the Faraday tensor (constitutive relations):

$$\star F_{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix} \ .$$

- **Gauss's law** and **Ampère's circuital law**,
  with Maxwell's correction,

$$\int_\Sigma \star F \quad \text{equals the } \textbf{sources} \ \Rightarrow \ \partial_\mu F^{\mu\nu} = J^\nu \ .$$

Newton's gravitation

- Gravitational **force field** $= m\boldsymbol{g}$.

- Field equations (and gravitational **potential**),

$$\left\{ \begin{array}{lll} \displaystyle\int_\gamma \boldsymbol{g}\cdot d\boldsymbol{\ell} = 0 & \Rightarrow & \boldsymbol{g} = -grad(\phi) \text{ and } curl(\boldsymbol{g}) = 0 \\ \displaystyle\int_\Sigma \boldsymbol{g}\cdot d\boldsymbol{a} = -4\pi\, m(\Sigma) & \Rightarrow & div(\boldsymbol{g}) = -4\pi\rho \end{array} \right. .$$

Maxwell's equations

$$\begin{cases} \displaystyle\oint_{\Sigma} \boldsymbol{B} \cdot d\boldsymbol{a} = 0 \\[2ex] \displaystyle\int_{\mathcal{S}} \frac{\partial}{\partial t}\boldsymbol{B} \cdot d\boldsymbol{a} + \oint_{\partial\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = 0 \\[2ex] \displaystyle\oint_{\Sigma} \boldsymbol{E} \cdot d\boldsymbol{a} = Q(\boldsymbol{\Sigma}) \\[2ex] \displaystyle\int_{\mathcal{S}} \frac{\partial}{\partial t}\boldsymbol{E} \cdot d\boldsymbol{a} - \oint_{\partial\mathcal{S}} \boldsymbol{B} \cdot d\boldsymbol{\ell} = -I(\partial\mathcal{S}) \end{cases}$$

How Maxwell constructed his electromagnetic theory

- He looked for a formalism able to explain all the **experimental facts** about electricity and magnetism, known at **his time**.

- He had to formulate a theory able to account for: **Gauss's law**, **Ampère's circuital law**, **Faraday's law of induction**, and the apparent absence of **magnetic poles**.

- In the course he **amended** Ampère's circuital law and provided a **better understanding** for phenomena involving **light**.

- Here I am performing a **similar task**: I am looking for a **classical formalism** able to explain all the **experimental facts** about electromagnetism known at **this time**.

The General Relativity analogy

- **Not** M and $\mathbf{g}$, but $(TM, \mathbf{g}, \nabla^{\mathbf{g}})$.

- **No** gravitational **force field**: only **tidal** forces, via $curv(\nabla^{\mathbf{g}})$.

$$\mathrm{Ric}(\nabla^{\mathbf{g}}) - \frac{1}{2}\mathrm{sc}(\nabla^{\mathbf{g}})\mathbf{g} = 8\pi\mathbf{T} \ .$$

- **No** potential (**no meaningful** vector calculus).

- Underlying **geometric** structure

$$(L, \langle\cdot,\cdot\rangle, \nabla) \ .$$

- **Equations** and **effects** via $curv(\nabla)$.

Another fact from nature: ES effect (magnetic AB effect)

- **Fields** $\boldsymbol{E}$ and $\boldsymbol{B}$ are **derived** from ∇ via its **curvature**.
  They are **no longer fundamental** objects of the theory.

- **Different** choices of connexions provide **inequivalent** curvatures (i.e. different fields $\boldsymbol{E}$ and $\boldsymbol{B}$).
  Contrary to potentials, which can provide the **same** fields for **different** choices.

- The **topologies** of both M and L are of relevance in the new formalism (**nonintegrable phases**).
  In **accordance** to the Ehrenberg–Siday effect.

A new framework for electromagnetism

- **Definition:** *an* ***electromagnetic field*** *over a spacetime region is a hermitian* ***line bundle*** *defined over the same spacetime region* ***excluding the portions occupied by sources*** *together with a hermitian* ***connexion*** *whose* ***curvature*** *satisfies that:*

  (i) *the integral of its Hodge dual over* ***any*** *gaussian surface* ***vanishes****,*

  (ii) *its integral over a gaussian surface multiplied by* $-e\sqrt{-1}$ *is the total* ***electric charge*** *contained in the gaussian surface,*

  (iii) *and its contraction with a unitary time vector field multiplied by* $e\sqrt{-1}$ *and integrated along an amperian loop is the total* ***electric current*** *passing through the amperian loop.*

In mathematical symbols

- Hermitian **connexion** and its **curvature**, ∇ and $curv(\nabla)$;

- **gaussian surface** Σ, **amperian loop** γ, and unitary **time vector** field X (e.g. $-\frac{\partial}{\partial t}$);

- *the* ***electromagnetic field tensor*** *is defined as* $\omega := -(e\sqrt{-1}) \cdot curv(\nabla)$, and one has:

(i) $\int_\Sigma \star\omega = 0$,

(ii) $\int_\Sigma \omega$ is the total **electric charge** inside Σ,

(iii) and $-\int_\gamma \imath_X \omega$ is the total **electric current** passing through γ.

Topological aspects of the sources

- **Theorem:** *Over* ***any*** *gaussian surface* Σ *in* $M - \{$*sources*$\}$

$$\int_\Sigma \sqrt{-1} \cdot curv(\nabla) \text{ is an } \textbf{integer},$$

and for gaussian surfaces ***satisfying*** $\Sigma = \partial\mathcal{V}$

$$\int_{\partial\mathcal{V}} \sqrt{-1} \cdot curv(\nabla) = 0 \ .$$

- $\partial\mathcal{V} = \Sigma' - \Sigma \ \Rightarrow \ \int_{\Sigma'} \omega = \int_\Sigma \omega \ .$

- $\partial\mathcal{S} = \gamma - \gamma' \ \Rightarrow \ -\int_\gamma \imath_X\omega = -\int_{\gamma'} \imath_X\omega +$ displacement current .

Comparison with Maxwell's theory: (ii) Gauss, and (iii) Ampère

- **Gauss's law**,

$$\int_{\Sigma} \omega = Q(\Sigma) \;\Rightarrow\; \oint_{\Sigma} \boldsymbol{E} \cdot d\boldsymbol{a} = Q(\Sigma) \ .$$

- **Ampère's circuital law**, with Maxwell's correction,

$$-\int_{\gamma} \imath_X \omega = I(\gamma) \;\Rightarrow\; \oint_{\gamma} \boldsymbol{B} \cdot d\boldsymbol{\ell} = I(\gamma') + \int_{\mathcal{S}} \frac{\partial}{\partial t} \boldsymbol{E} \cdot d\boldsymbol{a} \ .$$

- It holds $\omega = - \star F$.

Comparison with Maxwell's theory: (i) magnetic poles and Faraday

- **No** magnetic poles,

$$\int_{\Sigma} \star\omega = 0 \;\Rightarrow\; \oint_{\Sigma} \boldsymbol{B} \cdot d\boldsymbol{a} = 0 \ .$$

- **Faraday's law of induction**,

$$\int_{\Sigma} \star\omega = 0 \;\Rightarrow\; \int_{\mathcal{S}} \pounds_X(\star\omega) = \int_{\partial\mathcal{S}} \imath_X \star\omega \;\Leftrightarrow\; \int_{\mathcal{S}} \frac{\partial}{\partial t}\boldsymbol{B} \cdot d\boldsymbol{a} = -\oint_{\partial\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{\ell} \ .$$

- It holds $\star\omega = F$.

Simple nontrivial example on the Minkowski spacetime

- **Point charge** whose worldline is $\{x = y = z = 0\}$ has

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{E} = \frac{(q_e/4\pi)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Electric charge quantisation

- Hodge dual of a Faraday tensor for a **point charge** whose worldline is $\{x = y = z = 0\}$

$$\varpi_{\mu\nu} = \frac{(q_e/4\pi)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z & -y \\ 0 & -z & 0 & x \\ 0 & y & -x & 0 \end{pmatrix} \quad .$$

- If $\varpi = -(e\sqrt{-1}) \cdot curv(\nabla)$, then $\int_\Sigma \varpi / e$ is an integer.

$$\int_\Sigma \varpi = q_e \;\Rightarrow\; q_e = ke \ .$$

- **Theorem:** *the tensor ϖ is the electromagnetic* ***field tensor*** *of an electromagnetic field of a* ***point charge*** *whose wordline is $\{x = y = z = 0\}$* ***if and only if*** *q_e is an* ***integral*** *multiple of e.*

The Poisson picture of the Lorentz force law

- The hamiltonian **does not** change: **free particle**,

$$H(\boldsymbol{p}, \boldsymbol{x}) = \frac{1}{2m} \sum_{j=1}^{3} (p_j)^2 - (p_0)^2 \ .$$

- The classical commutation relations **do change**:

$$\{p_1, p_2\} = -qB_3 \ , \ \{p_1, p_3\} = qB_2 \ , \ \{p_2, p_3\} = -qB_1 \ ,$$

$$\{p_0, p_j\} = qE_j \ .$$

- Hamilton's equations are **equivalent** to Lorentz force law,

$$\begin{cases} \dot{\boldsymbol{p}} = \{\boldsymbol{p}, H\} \\ \dot{\boldsymbol{x}} = \{\boldsymbol{x}, H\} \end{cases} \ .$$