Drinfeld twists and Lorentz Symmetry Violation

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Outline

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Motivations

Field theory: fields (scalar, vector, tensor, etc.) are defined over a Riemannian manifolds. At very small distances this description of physical systems do not work.

Quantum mechanics: coordinates become operators with canonical commutation relations

$$[x_i, x_j] = [p_i, p_j] = 0, \quad [x_i, p_j] = i\delta_{ij}\hbar$$

Very high energies: or at even smaller distances the relation $[x_i, x_j] = 0$ can become inconsistent (Doplicher).

The notion of space-time as a Riemannian manifold **should be changed** for very small distances.

Candidates: to introduce a minimal possible distance; consider other topologies (not thin topologies); deform space-time, in particular, noncommutative models.

Noncommutativity via Drinfel'd twist deformations

In this talk we will consider noncommutativity in

- non-relativistic systems (for Abelian and Jordanian twists) and in
- relativistic systems (for Jordanian and Extended Jordanian twists)

For the space(time) coordinate operators the new commutators get the form

$$[x_i, x_j] = (something)$$

This "something" is obtained as a result of Drinfel'd twist deformation of universal enveloping algebra $\mathcal{U}(\mathcal{G})$ with a Hopf algebra structure, of some Lie algebra $\mathcal{G}$ which contains Heisenberg algebra $\mathcal{H} = \{\hbar, x_i, p_j\}$ as a subalgebra.

Notice that in this approach a Planck constant $\hbar$ should be considered as an **element of algebra** and **not as a multiple of identity**.

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Procedure:

1. Consider an initial (undeformed) quantum physical system (model, theory).
   Construct a **Hopf algebra structure** for this system in accordance with the physical properties of undeformed operators.
2. Apply Drinfel'd twist deformation on the operators of the quantum system: **deform Hopf algebra structure**.
3. Investigate the new properties of the deformed (in our case noncommutative) system.

Advantages:

- control of the deformation via deformation parameter;
- mathematical consistency of the procedure;
- simplicity and clearness of computations;
- control on multi-particle states;
- testability in experiment.

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Squeme of procedure:

(1. Hopf algebra A) $\rightarrow$ (2. Drifel'd twist $\mathcal{F}$) $\rightarrow$ (3. Deformed algebra $A^{\mathcal{F}}$)

... with deformed physical system associated to the deformed algebra $A^{\mathcal{F}}$

Where
1) is for the inicial (undeformed) system

2) the object $\mathcal{F}$ define a noncommutative transformation

3) is a system with noncommutative properties

Hopf algebras

Hopf algebra is an associative algebra A (over a field K) with extra operations defined:
coproduct $\Delta : A \to A \otimes A$,
counit $\varepsilon : A \to K$ and
antipode $S : a \to A$.

In physics coproduct is responsible for many-particle states.

If $\mathcal{O}$ is an operator related to an observable for one particle state.
2-particle state: $\Delta(\mathcal{O}) = \mathcal{O} \otimes Id + Id \otimes \mathcal{O}$
3-particle state: $(Id \oplus \Delta)(\mathcal{O}) = (\Delta \oplus Id)(\mathcal{O})$
$\dots$

Coproduct is also related with the Leibniz rule.

We consider a Lie algebra $\mathcal{G}$ and the **universal enveloping algebra $\mathcal{U}(\mathcal{G}) \equiv A$ as the associative algebra A** in the Hopf algebraic structure.

Generators g_i of $\mathcal{G}$ are called the **primitive elements.**

Coproduct
1) for primitive elements g_i

$$\Delta(g_i) = g_i \otimes \mathbf{1} + \mathbf{1} \otimes g_i$$

2) for the unit element $\mathbf{1}$ of $\mathcal{U}(\mathcal{G})$

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$$

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3) for other **(composite)** elements of $\mathcal{U}(\mathcal{G})$

$$\Delta(g_1 g_2) = \Delta(g_1)\Delta(g_2) = (\mathbf{1} \otimes g_1 + g_1 \otimes \mathbf{1})\,(\mathbf{1} \otimes g_2 + g_2 \otimes \mathbf{1})$$
$$= \mathbf{1} \otimes g_1 g_2 + g_1 g_2 \otimes \mathbf{1} + g_1 \otimes g_2 + g_2 \otimes g_1$$

Hense, in physical models when one apply Hopf algebraic structure, all additive observable operators should be primitive elements of $\mathcal{G}$

Example: energy of 2-particle state $E^{(2)} = E_1 + E_2$.
If Hamiltonian is a primitive element, it have to be expressed as $\Delta(H) = \mathbf{1} \otimes H + H \otimes \mathbf{1}$.

A Hopf algebraic structure to be physical, must contain in $\mathcal{G}$ all elements which correspond additive operators. This new $\mathcal{G}$ we call an **unfolded Lie algebra**.

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EXAMPLES of unfolded algebras:

1. Harmonic oscillator:

$$\mathcal{G} = \{x_i, p_i, \hbar, P_{ii}, X_{ii}, M_{ii}\},$$

   additional generators $X_{ii} = \frac{1}{\hbar}x_i^2$, $P_{ii} = \frac{1}{\hbar}p_i^2$, $M_{ii} = \frac{1}{\hbar}x_i p_i$.

2. Particle in constant electric $\vec{E}$ and magnetic B fields

$$\mathcal{G} = \{x_i, p_i, \hbar, P_{ii}, X_{ii}, M_{ij}\}$$

   additional generators of (1) and $M_{ij} = \frac{1}{\hbar}(x_i p_j + p_j x_i)$ $(i \neq j)$.

3. If at least one of potential terms in the Hamiltonian contains a term of power k ($k \geq 3$), the enlarged algebra is necessarily *infinite-dimensional*.
   Ex: anharmonic oscillator (x_i^4) or Coulomb $\frac{1}{r}$ potential.

Drinfeld twist $\mathcal{F}$

A deformation of a Hopf algebra is defined by Drinfeld twist

$$\mathcal{F} \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$$

which should satisfy the cocycle and counitarity conditions

1) $(\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F}$
responsible for the associativity of the deformed theory

2) $(\varepsilon \otimes id)\mathcal{F} = \mathbf{1} = (id \otimes \varepsilon)\mathcal{F}$
responsible for the fact that $\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + \ldots$
the inicial theory is recovered as a limit

Theory (Drinfel'd, Stolin, Kulish, Reshetikhin...):
the elements entering a Drinfel'd twist are taken from even-dimensional subalgebra of the Lie algebra $\mathcal{G}$.

The simplest case: 2-dimensional subalgebra. It can close in two ways:

1. $[a, b] = 0$ - defines so called Abelian twist

$$\mathcal{F} = \exp\left(i\alpha\epsilon_{ij} a \otimes b\right)$$

2. $[a, b] = \lambda b$ - defines Jordanian twist
   $\lambda \in K$

$$\mathcal{F} = \exp\left(-ia \otimes \sigma\right)$$

   where $\sigma = \ln(\mathbf{1} + \xi b)$.

   α and ξ are the deformation parameters. Undeformed theory is recovered for $\alpha \to 0$ ($\xi \to 0$)

Important formulas:

- Deformed primitive elements (new generators of Lie algeba):

$$g_i \mapsto g_i^{\mathcal{F}} = \bar{f}^{\alpha}(g_i)\bar{f}_{\alpha}$$

noncommutativity

- Deformed co-product

$$\Delta^{\mathcal{F}}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1}$$

new multiparticle states

- New multiplication law:

$$\left[m(f \otimes h) = fh\right] \to \left[m^{\mathcal{F}}(f \otimes h) = m(\mathcal{F}^{-1}(f \otimes h))\right]$$

Moyal product

It is interesting to analise physical consequences of **different twist deformations.**

and to see what twist can describe real physical systems

if noncommutativity is the solution for very high energies

So we apply different twists and see what happens.

This is the subject of the talk.

Abelian Twist

is defined as

$$\mathcal{F} = e^{i\alpha\epsilon_{ij}p_i \otimes p_j} \equiv f^{\beta} \otimes f_{\beta}$$

The twisted coordinate operators take the form

$$\begin{aligned} x_1^{\mathcal{F}} &= x_1 - \alpha\hbar p_2 \\ x_2^{\mathcal{F}} &= x_2 + \alpha\hbar p_1 \end{aligned}$$

The new nontrivial commutators are

$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = 2i\hbar^2\alpha.$$

The deformed product in space of functions is now a Moyal star-product. The **second order in $\hbar$** in coordinate commutators shows that in this model **NC appears in higher order in $\hbar$ ($\mathcal{O}(\hbar^2)$)** while the ordinary quantum effects are of the first ofder in $\hbar$ ($\mathcal{O}(\hbar)$).

Deformation

We consider NC deformation of the following operators ($d = 2$ for clearness in slides):

1. Square of radius:

$$R^2 = x_1^2 + x_2^2 = X_{11} + X_{22}$$

2. Harmonic oscillator Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega^2}{2}(x_1^2 + x_2^2) = \frac{1}{2}(P_{11} + P_{22}) + \frac{1}{2}\omega^2(X_{11} + X_{22})$$

3. Particle in constant electric and magnetic field:

$$H = \frac{1}{2m}(P_{11} + P_{22}) + \frac{m\omega_c^2}{8}(X_{11} + X_{22}) + \frac{\omega_c}{4}(M_{12} - M_{21}) + eEx_1$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequence.

Square distance operator

Twisted square distance operator

$$(R^2)^{\mathcal{F}} = X_{11} + X_{22} + 2\alpha(x_2 p_1 - x_1 p_2) + \alpha^2 \hbar (p_1^2 + p_2^2)$$

Can be rewritten in terms of criation/annihilation operators ($[b, b^\dagger] = \hbar^2$)

$$b = \frac{1}{2\sqrt{\alpha}}(x_1^{\mathcal{F}} + i x_2^{\mathcal{F}})$$

$$b^\dagger = \frac{1}{2\sqrt{\alpha}}(x_1^{\mathcal{F}} - i x_2^{\mathcal{F}})$$

$$(R^2)^{\mathcal{F}} = 2\alpha(bb^\dagger + b^\dagger b)$$

has a discrete spectrum with eigenvalues $4\alpha n_b + 2\alpha$ for $N_b = b^\dagger b$.

This result is in accordance with another description of noncommutativity given by F. G. Scholtz, L. Gouba, A. Hafver and C. M. Rohwer, J. Phys. A 42, 175303 (2009).

2-particle state

The 2-particle state operator $((\Omega)^{\mathcal{F}})^{(2)}$ is obtained from $\Delta(\Omega^{\mathcal{F}})$.

$$\begin{aligned}\Omega^{\mathcal{F}} &= a(P^{11}+P^{22})+b(X^{11}+X^{22})+c(M^{12}-M^{21})+dx_1\\ &+ \alpha\left[2b(x_2p_1-x_1p_2)-2c(p_1^2+p_2^2)-d\hbar p_2\right]\\ &+ \alpha^2 b\hbar(p_1^2+p_2^2)\end{aligned}$$

Defining Ω_0, Ω_1 and Ω_2 as

$$\Omega^{\mathcal{F}} \equiv \Omega_0+\alpha\Omega_1+\alpha^2\Omega_2$$

we have
$\Delta(\Omega^{\mathcal{F}}) = \Delta(\Omega_0)+2\alpha b\Delta(x_2p_1-x_1p_2)-2\alpha c\Delta(p_1^2+p_2^2)$
$-\alpha d\Delta(\hbar p_2)+\alpha^2 b\Delta[\hbar(p_2^2+p_2^2)]$

The resulting 2-particle operator splits into:

$$\begin{aligned}(\Omega^{\mathcal{F}})^{(2)} &\equiv \Delta(\Omega^{\mathcal{F}}) \\ &= \Omega^{\mathcal{F}} \otimes 1 + 1 \otimes \Omega^{\mathcal{F}} + \Omega_r \otimes 1 + 1 \otimes \Omega_r + \hat{\Omega}_{mixed}\end{aligned}$$

where
$\Omega_r = -\alpha d p_2 + \alpha^2 b(p_1^2 + p_2^2)$ and
$\hat{\Omega}_{mixed} = -2\alpha b \epsilon_{ij}(x_i \otimes p_j + p_j \otimes x_i) + 4(-\alpha c + \alpha^2 b)(p_1 \otimes p_1 + p_2 \otimes p_2)$
The last term includes contributions from both particles.

"Fundamental" and twisted NC, 2-particle state

Let us consider the case (II) (Harmonic oscillator) and let $m = 1$.

Defining center of mass coordinates: $x_i^{c.m.} = 1/2(x_i^{(1)} + x_i^{(2)})$,

momenta $p_i^{c.m.} = p_i^{(1)} + p_i^{(2)}$ $(i = 1, 2)$,

relative coordinates: $x_i^{rel} = 1/2(x_i^{(2)} - x_i^{(1)})$,

and relative momenta $p_i^{rel} = p_i^{(2)} - p_i^{(1)}$.

- **(A)** Undeformed Hamiltonian (harmonic oscillator):
  $H = \frac{1}{2}(p_i^{c.m.})^2 + 2\omega^2(x_i^{c.m.})^2 + \frac{1}{2}(p_i^{rel})^2 + 2\omega^2(x_i^{rel})^2$
- **(B)** "Fundamental" NC Hamiltonian:
  $H = \frac{1}{2}(1 + \alpha^2\omega^2)(p_i^{c.m.})^2 + 2\omega^2(x_i^{c.m.})^2 - 2\alpha^2\omega^2\epsilon_{ij}(x_i^{c.m.}p_j^{c.m})$
  $+\frac{1}{2}(1 + \alpha^2\omega^2)(p_i^{rel})^2 + 2\omega^2(x_i^{rel})^2 - 2\alpha\omega^2\epsilon_{ij}x_i^{rel}p_j^{rel}$
- **(C)** Twist deformed NC Hamiltonian:
  $H = (\frac{1}{2} + 2\alpha^2\omega^2)(p_i^{c.m.})^2 + 2\omega^2(x_i^{c.m.})^2 - 4\alpha\omega^2\epsilon_{ij}(x_i^{c.m.}p_j^{c.m.})$
  $+\frac{1}{2}(p_i^{rel})^2 + 2\omega^2(x_i^{rel})^2$
  In the 2-particle Hamiltonian obtained via twist, contrary to the "fundamental" NC, the deformation appears only in the center of mass dynamics.

Energy eigenvalues in 2-particle case

- **(A)** Undeformed Hamiltonian:

$$E_{12} = 2\omega(n_1 + n_2) = 4\omega$$

- **(B)** ”Fundamental” NC Hamiltonian:

$$E_{12} = 2\omega\sqrt{1+\alpha^2\omega^2}(n_1 + n_2 + 2) + 2\alpha\omega^2(j_1 + j_2)$$

- **(C)** Twist deformed NC Hamiltonian:

$$E_{12} = 2\omega\sqrt{1+4\alpha^2\omega^2}(n_1 + 1) + 2\omega(n_2 + 1) + 4\alpha\omega^2 j_1$$

Here n_1 is associated with the center of mass coordinates while n_2 is asoociated with the relative coordinates and $j = -n, -n+2, \ldots, n-2, n$.

Energy levels

"fundamental" NC	$4\omega\sqrt{1+\alpha^2\omega^2}$ $6\omega\sqrt{1+\alpha^2\omega^2} - 2\alpha\omega^2$ $6\omega\sqrt{1+\alpha^2\omega^2} + 2\alpha\omega^2$ $8\omega\sqrt{1+\alpha^2\omega^2} - 4\alpha\omega^2$ $8\omega\sqrt{1+\alpha^2\omega^2}$ $8\omega\sqrt{1+\alpha^2\omega^2} + 4\alpha\omega^2$ $\ldots$
"twist" NC	$4\omega + 2\omega\sqrt{1+4\alpha^2\omega^2}$ $4\omega\sqrt{1+4\alpha^2\omega^2} \mp 4\alpha\omega^2$ $6\omega + 2\omega\sqrt{1+4\alpha^2\omega^2}$ $4\omega + 4\omega\sqrt{1+4\alpha^2\omega^2} \mp 4\alpha\omega^2$ $2\omega + 6\omega\sqrt{1+4\alpha^2\omega^2} - 8\alpha\omega^2$ $2\omega + 6\omega\sqrt{1+4\alpha^2\omega^2}$ $2\omega + 6\omega\sqrt{1+4\alpha^2\omega^2} + 8\alpha\omega^2$ $\ldots$

Jordanian twist

Case $[a, b] = \lambda b$.

I): $sl(2)$ **Lie algebra** with generators $\{D, H, K\}$

$$\begin{aligned} [D, H] &= H \\ [D, K] &= -K \\ [K, H] &= 2D \end{aligned}$$

we use Borel subalgebra spanned by D and H

Jordanian deformation of $sl(2)$ is obtained from the twist

$$\mathcal{F} = \exp\left(-iD \times \sigma\right)$$

where $\sigma = \ln(\mathbf{1} + \xi H)$, and H, D, K are $sl(2)$ generators.

Dubois-Violette and Launer (1990), Ohn (1992), Ogievetsky (1993), Kulish and Celeghini (1998), Boroviec, Lukierski and Tolstoy (2003).

II): 2-dimensional subalgebra of Poincare algebra

$$[P_0, P_1] = iC,$$
$$[M_{01}, P_0] = -iP_1,$$
$$[M_{01}, P_1] = iP_0.$$

For the Jordanian twist we take $a \equiv -M_{01}$ and $b \equiv P_0 + P_1$

III): 2k-dimensional subalgebra of Poincare algebra

$$\mathcal{F} = \exp\left(iM \otimes \ln(1 + \rho P_+) + i\epsilon M_{+j} \otimes \ln(1 + \rho P_+) \frac{P_j}{P_+} \right).$$

extended Jordanian twist

$M = M_{01}$, $P_+ = P_0 + P_1$,

Deformation of $sl(2)$

The twist induces a deformation $g \mapsto g^{\mathcal{F}}$.

The deformed generators are given by

$$\begin{aligned} x_i^{\mathcal{F}} &= x_i e^{\frac{\sigma}{2}}, \quad p_i^{\mathcal{F}} = p_i e^{-\frac{\sigma}{2}}, \quad \hbar^{\mathcal{F}} = \hbar \\ H^{\mathcal{F}} &= He^{-\sigma}, \quad K^{\mathcal{F}} = Ke^{\sigma}, \quad D^{\mathcal{F}} = D. \end{aligned}$$

The commutator of the deformed position variables has the form:

$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = -\frac{i\xi}{2}\left(x_i^{\mathcal{F}} p_j^{\mathcal{F}} - x_j^{\mathcal{F}} p_i^{\mathcal{F}}\right) + \mathcal{O}(\xi^3).$$

This is the noncommutativity considered by Snyder in H. S. Snyder, Phys. Rev. **71** (1947) 38.

Pseudo-Hermiticity of the Hamiltonian

$$
\begin{aligned}
\mathbf{H} &= H_\rho + (\bar{\rho} - \rho)K^{-1} + \lambda K, \\
\mathbf{H}^{\mathcal{F}} &= H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}} + \lambda K^{\mathcal{F}} \\
&= H_\rho T^{-2} + (\bar{g} - g)K^{-1}T^{-2} + \lambda K T^2
\end{aligned}
$$

where $T = e^{\frac{\sigma}{2}}$.

Two types of η-hermiticity:

1. $\bar{\rho} - \rho \neq 0$, $\lambda = 0$ (Calogero type)
   $\left[H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}}\right]^\dagger = \eta \left(H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}}\right) \eta^{-1}$
   with $\eta = T^{-2}$,
2. $\bar{\rho} - \rho = 0$, $\lambda \neq 0$ (harmonic oscillator)
   $\left[H_\rho^{\mathcal{F}} + \lambda K^{\mathcal{F}}\right]^\dagger = \eta \left(H_\rho^{\mathcal{F}} + \lambda K^{\mathcal{F}}\right) \eta^{-1}$ with $\eta = T^2$

Jordanian deformations of Poincaré

- 1998, Kulish, Liakhovski, Mudrov, Extended jordanian twists for Lie algebras - introduced the first time for $sl(N)$.
- from 1998, Kulish, Reshetikhin, Boroviec, Lukierski, Tolstoi - generazised for superalgebras ($osp(1|2) \ldots osp(1|2n)$ and $sl(m|n)$)
- physical applications (Poincare, spin chains, κ-Minkowski, Newton-Hooke, AdS and others)

The non-abelian algebra *ii* induces the (jordanian) twist

$$\mathcal{F} \quad = \quad \exp(-ia \otimes \ln(1+\xi b)),$$

where ξ is the (dimensional) deformation parameter.

Under the transposition operator $\tau(v \otimes w) = w \otimes v$, the transposed twist

$$\mathcal{F}_\tau \quad := \quad \exp(-i \ln(1+\rho b) \otimes a),$$

still satisfies the cocycle condition.

The jordanian and extended jordanian twist of the d-dimensional Poincaré algebra are

$$\mathcal{F} = \exp\left(iM\otimes \ln(1+\rho P_+) + i\epsilon M_{+j}\otimes \ln(1+\rho P_+)\frac{P_j}{P_+}\right).$$

The jordanian case is recovered for $\epsilon = 0$;
the extended jordanian case is recovered for $\epsilon = 1$.

Under transposition, the $\mathcal{F}_\tau$ twists are

$$\mathcal{F}_\tau = \exp\left(i\ln(1+\rho P_+)\otimes M + i\epsilon \ln(1+\rho P_+)\frac{P_j}{P_+}\otimes M_{+j}\right).$$

Under twist-deformation, a generator $g\in\mathcal{G}$ is mapped into the Universal Enveloping Algebra element $g^{\mathcal{F}}\in\mathcal{U}(\mathcal{G})$, given by

$$g^{\mathcal{F}} = \overline{f}^{\alpha}(g)\overline{f}_{\alpha}, \quad (\mathcal{F}^{-1} = \overline{f}^{\alpha}\otimes\overline{f}_{\alpha}).$$

Four twist-deformations: the jordanian deformations ($\epsilon = 0$) based on $\mathcal{F}$ (case *I*) and $\mathcal{F}_\tau$ (case *II*) and the extended jordanian deformations ($\epsilon = 1$) based on $\mathcal{F}$ (case *III*) and $\mathcal{F}_\tau$ (case *IV*).

Case I

Basis of twist-deformed generators:

$$
\begin{aligned}
P_+^{\mathcal{F}} &= P_+ \frac{1}{1+\rho P_+}, \\
P_-^{\mathcal{F}} &= P_-(1+\rho P_+), \\
P_j^{\mathcal{F}} &= P_j, \\
M^{\mathcal{F}} &= M, \\
M_{+j}^{\mathcal{F}} &= M_{+j} \frac{1}{1+\rho P_+}, \\
M_{-j}^{\mathcal{F}} &= M_{-j}(1+\rho P_+), \\
N^{\mathcal{F}} &= N
\end{aligned}
$$

$$
\begin{aligned}
x_+^{\mathcal{F}} &= x_+ \frac{1}{1+\rho P_+}, \\
x_-^{\mathcal{F}} &= x_-(1+\rho P_+), \\
x_j^{\mathcal{F}} &= x_j.
\end{aligned}
$$

The undeformed generators can be expressed in terms of the deformed generators on the basis of inverse formulas.

$\mathcal{F}_\tau$ (case II, IV) twist-generator basis:

$$
\begin{aligned}
P_\bullet^{\mathcal{F}} &= P_\bullet, \\
M^{\mathcal{F}} &= \frac{1+2\rho P_+}{1+\rho P_+} M + \epsilon \left(\frac{\rho P_j}{1+\rho P_+} - \ln(1+\rho P_+) \frac{P_j}{P_+} \right) M, \\
M_{+j}^{\mathcal{F}} &= M_{+j} + \epsilon \ln(1+\rho P_+) M_{+j}, \\
M_{-j}^{\mathcal{F}} &= M_{-j} + \frac{2\rho P_j}{1+\rho P_+} M + \\
&\quad \epsilon \left(\ln(1+\rho P_+) \frac{P_j}{P_+} \delta_{jk} + \frac{2\rho P_j P_k}{(1+\rho P_+) P_+} - 2 \ln(1+\rho P_+) \frac{P_j P_k}{{P_+}^2} \right) \\
N^{\mathcal{F}} &= N - \epsilon \epsilon_{jk} \ln(1+\rho P_+) \frac{P_k}{P_+} M_{+j}
\end{aligned}
$$

$$
\begin{aligned}
x_+^{\mathcal{F}} &= x_+, \\
x_-^{\mathcal{F}} &= x_- + \frac{2\rho\hbar}{1+\rho P_+} M + 2\epsilon\hbar \left(\frac{\rho}{1+\rho P_+} - \frac{\ln(1+\rho P_+)}{P_+} \right) \frac{P_j}{P_+} M_{+j}, \\
x_j^{\mathcal{F}} &= x_j - \epsilon \epsilon_{jk} \hbar \ln(1+\rho P_+) \frac{1}{P_+} M_{+j}.
\end{aligned}
$$

For an operator Ω the hermiticity condition is $\Omega^\dagger = \Omega$.
The pseudohermiticity condition is $\Omega^\dagger = \eta\Omega\eta^{-1}$, for an invertible hermitian operator $\eta = \eta^\dagger$.
Case *I* (jordanian $\mathcal{F}$ twist with $\epsilon = 0$):

$$
\begin{aligned}
P_\bullet^{\mathcal{F}\dagger} &= \eta^\lambda P_\bullet^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R},\\
M^{\mathcal{F}\dagger} &= M^{\mathcal{F}}, \qquad i.e.\ \lambda = 0,\\
M_{+j}^{\mathcal{F}\dagger} &= \eta^\lambda M_{+j}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R},\\
M_{-j}^{\mathcal{F}\dagger} &= \eta M_{-j}^{\mathcal{F}} \eta^{-1}, \quad i.e.\ \lambda = 1,\\
N^{\mathcal{F}\dagger} &= \eta^\lambda N^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R},
\end{aligned}
$$

$$
\begin{aligned}
x_+^{\mathcal{F}\dagger} &= \eta^\lambda x_+^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R},\\
x_-^{\mathcal{F}\dagger} &= \eta x_-^{\mathcal{F}} \eta^{-1}, \quad i.e.\ \lambda = 1,\\
x_j^{\mathcal{F}\dagger} &= \eta^\lambda x_j^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R},
\end{aligned}
$$

for the hermitian $\eta = 1 + \rho P_+$.
The deformed subalgebra contains a subalgebra of hermitian operators (for $\lambda = 0$). The operators $M_{-j}^{\mathcal{F}}, x_-^{\mathcal{F}}$ are not hermitian.

Case *II* (jordanian $\mathcal{F}_\tau$ twist with $\epsilon = 0$):

$$
\begin{aligned}
P_\bullet^{\mathcal{F}\dagger} &= \eta^\lambda P_\bullet^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\
M^{\mathcal{F}\dagger} &= \eta M^{\mathcal{F}} \eta^{-1}, \quad i.e.\ \lambda = 1, \\
M_{+j}^{\mathcal{F}\dagger} &= \eta^\lambda M_{+j}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\
N^{\mathcal{F}\dagger} &= \eta^\lambda N^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R},
\end{aligned}
$$

$$
\begin{aligned}
x_+^{\mathcal{F}\dagger} &= \eta^\lambda x_+^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R}, \\
x_j^{\mathcal{F}\dagger} &= \eta^\lambda x_j^{\mathcal{F}} \eta^{-\lambda}, \quad \forall \lambda \in \mathbb{R},
\end{aligned}
$$

for the hermitian $\eta = \frac{1+\rho P_+}{1+2\rho P_+}$.
The deformed generators $M_{-j}^{\mathcal{F}}, x_-^{\mathcal{F}}$ do not have nice (pseudo)-hermiticity properties as the previous generators.
The deformed subalgebra contains a subalgebra of pseudo-hermitian operators for the common choice $\lambda = 1$.

Deformation as a non-linear W-algebra (case I):

$$\mathcal{P}_{drf} \ : \ \{P^{\mathcal{F}}_{\pm}, P^{\mathcal{F}}_{j}, M^{\mathcal{F}}, N^{\mathcal{F}}, M^{\mathcal{F}}_{\pm j}\},$$

$$\mathcal{P}_{obs} \ : \ \{P^{\mathcal{F}}_{\pm}, P^{\mathcal{F}}_{j}, M^{\mathcal{F}}, N^{\mathcal{F}}, M^{\mathcal{F}}_{+j}\}.$$

$$\begin{aligned}
[P^{\mathcal{F}}_{\pm}, M^{\mathcal{F}}] &= \pm i P^{\mathcal{F}}_{\pm} \mp i\rho P^{\mathcal{F}}_{+} P^{\mathcal{F}}_{\pm}, \\
[P^{\mathcal{F}}_{+}, M^{\mathcal{F}}_{+j}] &= 2iP^{\mathcal{F}}_{j} - 2i\rho P^{\mathcal{F}}_{+} P^{\mathcal{F}}_{j}, \\
[P^{\mathcal{F}}_{-}, M^{\mathcal{F}}_{+j}] &= 2iP^{\mathcal{F}}_{j}, \\
[P^{\mathcal{F}}_{-}, M^{\mathcal{F}}_{-j}] &= 2i\rho P^{\mathcal{F}}_{-} P^{\mathcal{F}}_{j}, \\
[P^{\mathcal{F}}_{j}, M^{\mathcal{F}}_{\pm k}] &= 2i\delta_{jk} P^{\mathcal{F}}_{\pm}, \\
[M^{\mathcal{F}}, M^{\mathcal{F}}_{\pm j}] &= \mp i M^{\mathcal{F}}_{\pm j} \pm i\rho M^{\mathcal{F}}_{\pm j} P^{\mathcal{F}}_{+}, \\
[N^{\mathcal{F}}, M^{\mathcal{F}}_{\pm j}] &= i\epsilon_{jk} M^{\mathcal{F}}_{\pm k}, \\
[M^{\mathcal{F}}_{+j}, M^{\mathcal{F}}_{-k}] &= -2i\delta_{jk} M^{\mathcal{F}} - 2i\epsilon_{jk} N^{\mathcal{F}} - 2i\rho M^{\mathcal{F}}_{+j} P^{\mathcal{F}}_{k}, \\
[M^{\mathcal{F}}_{-j}, M^{\mathcal{F}}_{-k}] &= 2i\rho(M^{\mathcal{F}}_{j} P^{\mathcal{F}}_{k} - M^{\mathcal{F}}_{-k} P^{\mathcal{F}}_{j}),
\end{aligned}$$

Deformed system, case I

Let $P_j \equiv 0$, massive particle with $m = 1$.
After the twist

$$\begin{aligned} P_+^{\mathcal{F}} &= \frac{P_+}{1+\xi P_+}, \\ P_-^{\mathcal{F}} &= P_-(1+\xi P_+). \end{aligned}$$

The condition $\xi \geq 0$ has to be imposed to avoid singularities.
On-shell we have

$$\begin{aligned} P_-^{\mathcal{F}} &= \frac{1}{P_+}(1+\xi P_+), \\ P_0^{\mathcal{F}} &= \frac{1}{2}\left(\frac{P_+}{1+\xi P_+} + \frac{1}{P_+}(1+\xi P_+)\right), \\ P_1^{\mathcal{F}} &= \frac{1}{2}\left(\frac{P_+}{1+\xi P_+} - \frac{1}{P_+}(1+\xi P_+)\right). \end{aligned}$$

The rest condition $P_1^{\mathcal{F}} = 0$ for the deformed P_1 momentum is obtained for

$$P_+ = \frac{1}{1-\xi}.$$

It can only be obtained for $\xi < 1$.
The range of the deformation parameter ξ is

$$0 \leq \xi < 1.$$

The range of the deformed operators:

$$\begin{aligned} P_+^{\mathcal{F}} &\in]0, \frac{1}{\xi}[, \\ P_-^{\mathcal{F}} &\in]\xi, +\infty], \\ P_0^{\mathcal{F}} &\in [1, +\infty], \\ P_1^{\mathcal{F}} &\in [-\infty, \frac{1}{2}(\xi - \frac{1}{\xi})[. \end{aligned}$$

$\frac{1}{\xi}$ can be interpreted as the maximal admissible P_+ momentum.

2-particle effects

Induced by the coproduct, the 2-particle addition formula for the deformed P_+ momenta reads as follows

$$(P_+^{\mathcal{F}})_{1+2} = \frac{(P_+^{\mathcal{F}})_1 + (P_+^{\mathcal{F}})_2}{1 + \xi((P_+^{\mathcal{F}})_1 + (P_+^{\mathcal{F}})_2)}.$$

let $P_+ \equiv x$

Closely expressed in terms of the deformed P_+ momenta we obtain the non-linear addition formula

$$\overline{x}_{1+2} = \frac{\overline{x}_1 + \overline{x}_2 - 2\xi\overline{x}_1\overline{x}_2}{1 - \xi^2\overline{x}_1\overline{x}_2}.$$

It is useful to compare this formula with the non-linear addition of velocities in special relativity. Let us change variables once more and set $\overline{x}_{1,2} = v_{1,2}$, $z = \frac{1}{c}$.
In special relativity we get

$$v_{1+2,s.r.} = \frac{v_1 + v_2}{1 + \frac{1}{c^2} v_1 v_2}.$$

In the above jordanian deformation we have

$$v_{1+2,jd.} = \frac{v_1 + v_2 - \frac{2}{c} v_1 v_2}{1 - \frac{1}{c^2} v_1 v_2}.$$

Let us compare their properties. Both non-linear addition formulas are symmetric in the $v_1 \leftrightarrow v_2$ exchange. They are both associative. They can also be defined in the box $0 \leq v_{1,2} \leq c$, so that the non-linear additive velocities belong to the $[0, c]$ range (in both cases if $v_1 = 0$, then $v_{1+2} = v_2$ and, if $v_1 = c$, $v_{1+2} = c$).
The main difference is that in special relativity the formula can be nicely extended to negative velocities belonging to the $-c \leq v_{1,2} \leq c$ box.

Thank you for the attention