On excited states in real-time AdS/CFT

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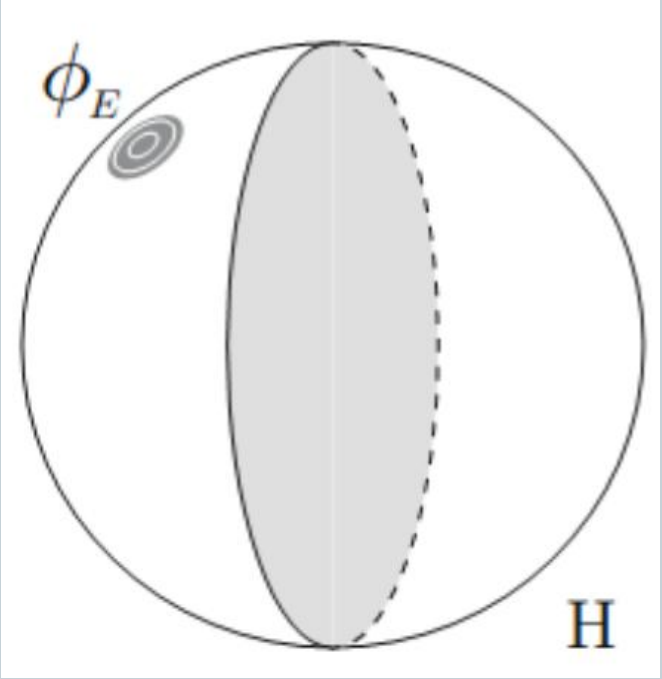
Motivation

- GKPW is an Euclidean prescription to calculate n-point functions of CFT local operators through calculations in the dual bulk theory.

$$\langle 0| e^{\int_{\partial H} \mathcal{O}\phi_E} |0\rangle \equiv \int \mathcal{D}\Phi_{[\Phi|_{\partial_H}=\phi_E]} e^{-S_E[\Phi]}$$

$$\Big\downarrow \text{Large N}$$

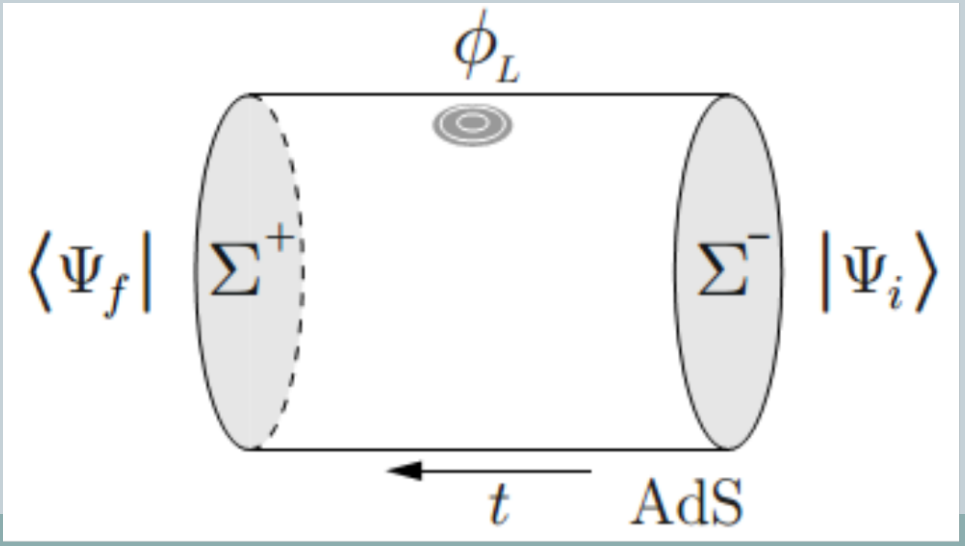
$$\langle 0| e^{\int_{\partial H} \mathcal{O}\phi_E} |0\rangle \equiv e^{-S^0_E[\phi_E]}$$



Motivation

- The Skenderis and van Rees prescription is a real-time extension of the GKPW prescription.
  - However, in Lorentzian signature, initial and final conditions are required in order to have a well defined problem.
  - These conditions should be related to the initial and final states, but so far no explicit interpretation in the dual CFT has been given.

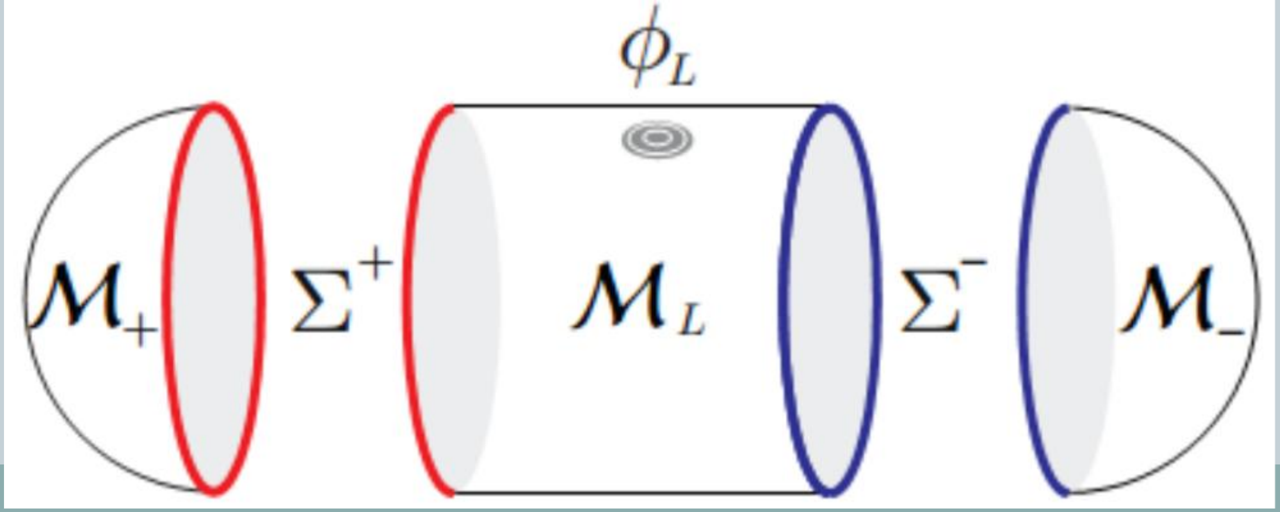
$$\phi_{\Sigma^\pm} \overset{?}{\longrightarrow} |\Psi_{i/f}\rangle$$



Motivation

- The Skenderis and van Rees prescription is a real time extension of the GKPW prescription.
  - The authors demonstrated that one can obtain time ordered n-point functions in the vacuum operator by gluing Lorentzian and Euclidean regions, with Dirichlet boundary conditions.

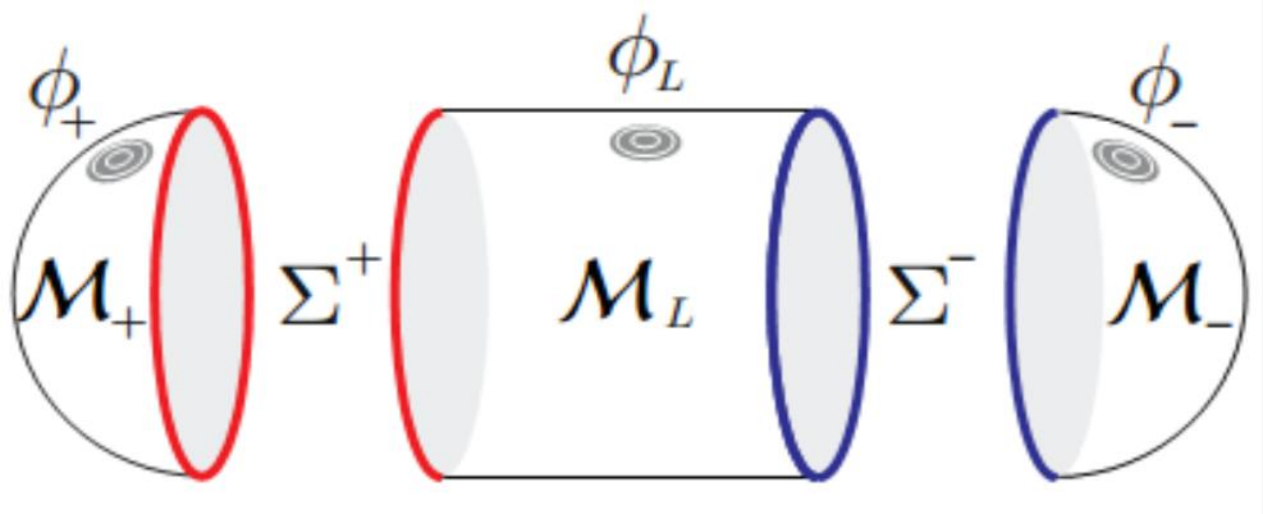
$$\langle 0|T[e^{-i\int_{\partial_r \mathcal{M}_L}\mathcal{O}\,\phi_L}]|0\rangle \equiv e^{-S^0_-[0;\phi_{\Sigma-}]+iS^0_L[\phi_L;\phi_{\Sigma-},\phi_{\Sigma+}]-S^0_+[0;\phi_{\Sigma+}]}$$



Motivation

- The natural thing to do next is to turn on sources $\phi^{\pm}$ on $\mathcal{M}_{\pm}$ and see whether we reach excited states and precisely which states are we describing.

$$\langle\phi_+|T[e^{-i\int_{\partial_r\mathcal{M}_L}\mathcal{O}\,\phi_L}]|\phi_-\rangle \equiv e^{-S^0_-[\phi_-;\phi_{\Sigma-}]+iS^0_L[\phi_L;\phi_{\Sigma-},\phi_{\Sigma+}]-S^0_+[\phi_+;\phi_{\Sigma+}]}$$



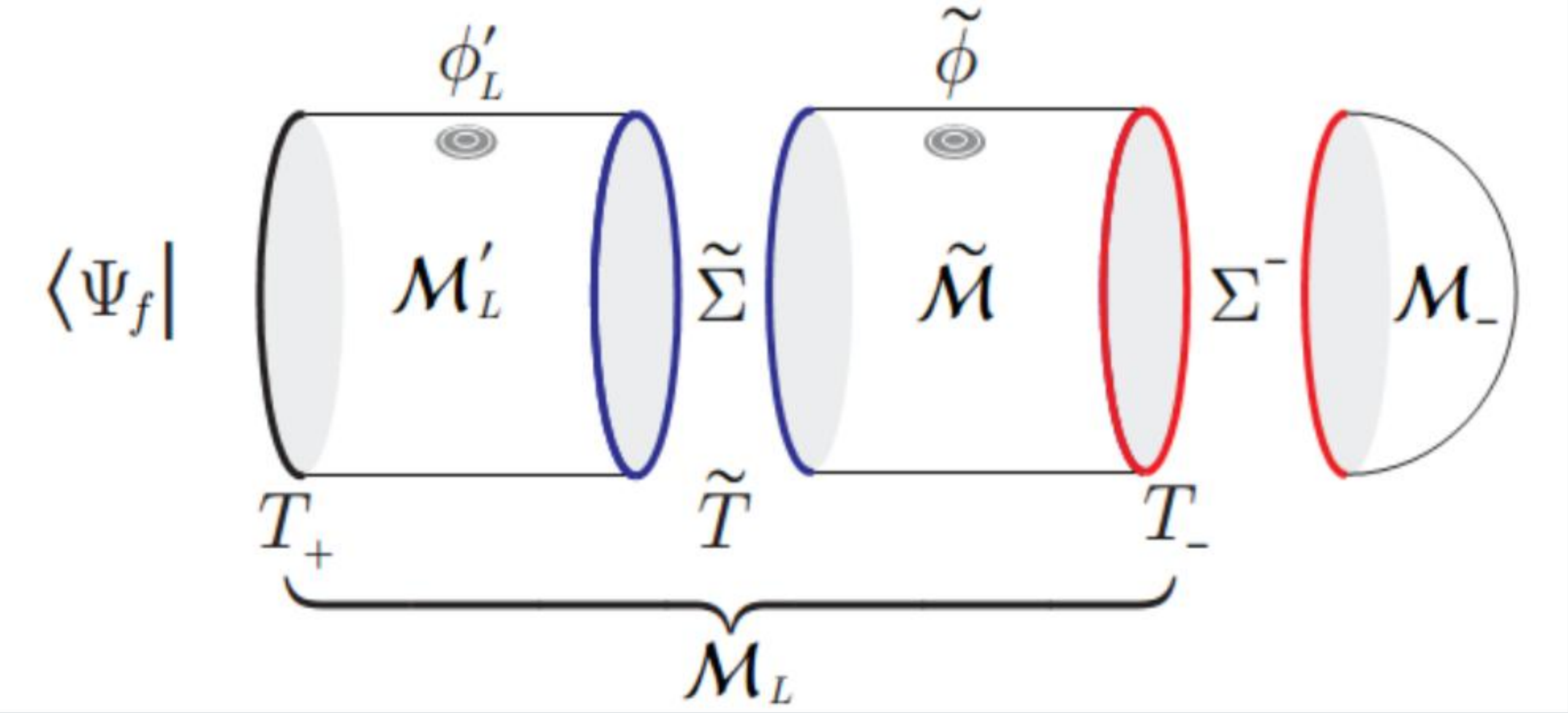
¿Excited States?

- But before studying a concrete example, can we say something general about these states?

It turns out we can!

Excited States

- In order to unravel the nature of the states $|\phi_\pm\rangle$ one can consider the following set-up:



WARNING! FORMALISM AHEAD!

- Even if SvR stated their prescription as

$$\langle 0|T[e^{-i\int_{\partial_r \mathcal{M}_L} \mathcal{O}\,\phi_L}]|0\rangle \equiv e^{-S^0_-[0;\phi_{\Sigma-}]+iS^0_L[\phi_L;\phi_{\Sigma-},\phi_{\Sigma+}]-S^0_+[0;\phi_{\Sigma+}]}$$

one can safely assume (and they hint towards it) that it comes from a saddle point approximation of

$$\langle 0|\,T[e^{-i\int_{\partial_r \mathcal{M}_L} \mathcal{O}\phi_L}]\,|0\rangle \equiv \langle \Psi^0\,|\,\mathcal{U}[T_-,T_+]_{\phi_L}\,|\,\Psi^0\rangle$$

$$= \sum_{\phi_{\Sigma\pm}} \left(\Psi_0[\phi_{\Sigma+}]\,\right)^* \,\mathcal{Z}[\phi_L;\phi_{\Sigma-},\phi_{\Sigma+}]\,\Psi_0[\phi_{\Sigma-}]$$

Where the Lorentzian part is by definition

$$\mathcal{Z}[\phi_L;\phi_{\Sigma-},\phi_{\Sigma+}] \equiv \int_{\phi_{\Sigma-}}^{\phi_{\Sigma+}} [\mathcal{D}\Phi]_{\phi_L}\; e^{iS_L[\Phi]}$$

and one should recall that Hartle & Hawking taught us that

$$\Psi^0[\phi_{\Sigma-}] \equiv \int_0^{\phi_{\Sigma-}} [\mathcal{D}\Phi]_0\, e^{-S_-[\Phi]}$$

WARNING! FORMALISM AHEAD!

The relevant part of this is that one can rewrite SvR as

$$\langle 0|\, T[e^{-i\int_{\partial_r \mathcal{M}_L} \mathcal{O}\phi_L}]\,|0\rangle \equiv \sum_{\phi_{\Sigma\pm}} \left(\Psi_0[\phi_{\Sigma+}]\,\right)^* \; \mathcal{Z}[\phi_L;\phi_{\Sigma-},\phi_{\Sigma+}]\,\Psi_0[\phi_{\Sigma-}]$$

$$= \sum_{\phi_{\Sigma\pm}} \left(\int_{\phi_{\Sigma+}}^{0} [\mathcal{D}\Phi]_0\, e^{-S_+[\Phi]}\right) \left(\int_{\phi_{\Sigma-}}^{\phi_{\Sigma+}} [\mathcal{D}\Phi]_{\phi_L}\, e^{iS_L[\Phi]}\right) \left(\int_0^{\phi_{\Sigma-}} [\mathcal{D}\Phi]_0\, e^{-S_-[\Phi]}\right)$$

and that for a arbitrary final state

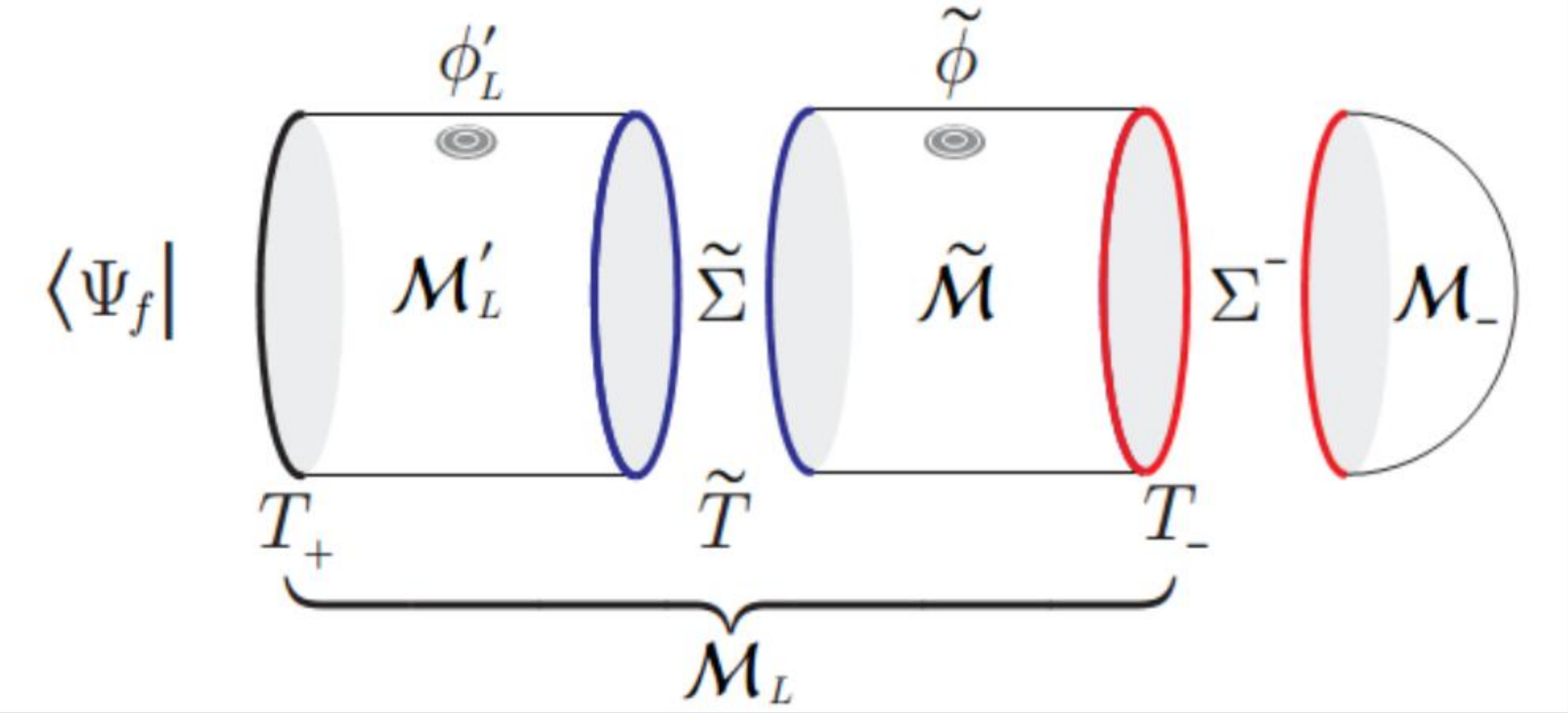
$$\langle \Psi_f|\, T[e^{-i\int_{\partial_r \mathcal{M}_L} \mathcal{O}\phi_L}]\,|0\rangle \equiv \sum_{\phi_{\Sigma\pm}} \left(\Psi_f[\phi_{\Sigma+}]\,\right)^* \left(\int_{\phi_{\Sigma-}}^{\phi_{\Sigma+}} [\mathcal{D}\Phi]_{\phi_L}\, e^{iS_L[\Phi]}\right) \times$$

$$\times \left(\int_0^{\phi_{\Sigma-}} [\mathcal{D}\Phi]_0\, e^{-S_-[\Phi]}\right)$$

which we will use in the following argument.

Excited States

- In order to unravel the nature of the states $|\phi_{\pm}\rangle$ one can consider the following set-up:



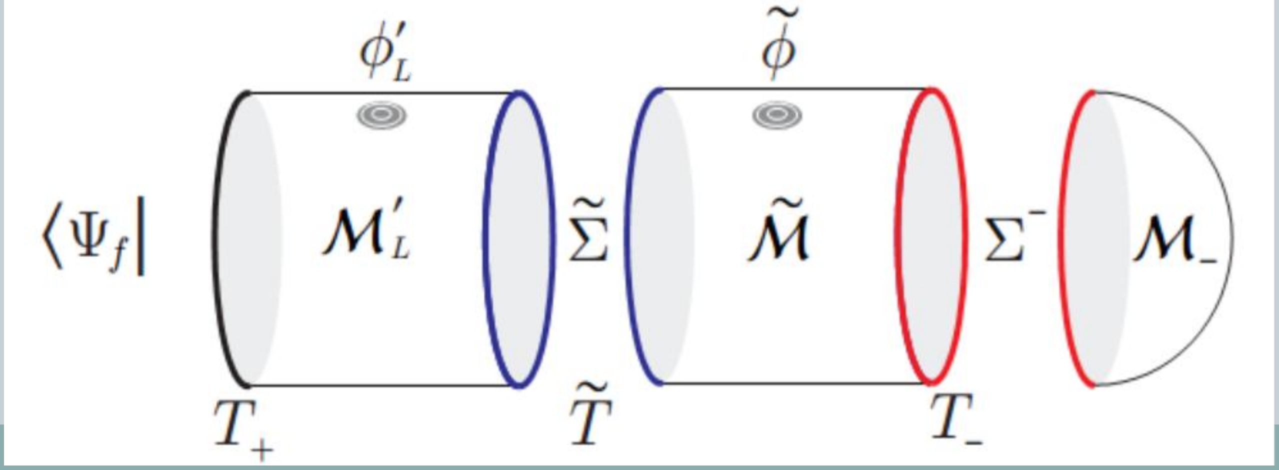
Excited States

- We will Wick-rotate $\tilde{\mathcal{M}}$ such that

$$\langle \Psi_f | T[e^{-i\int_{\partial_r \mathcal{M}'_L} \mathcal{O}\,\phi'_L \; -i\int_{\partial_r \tilde{\mathcal{M}}} \mathcal{O}\,\tilde{\phi}}] |0\rangle \Longrightarrow \langle \Psi_f | T[e^{-i\int_{\partial_r \mathcal{M}'_L} \mathcal{O}\,\phi'_L}] \left(e^{-\int_{\partial_r \tilde{\mathcal{M}}} \mathcal{O}\,\tilde{\phi}} \,|0\rangle \right)$$

and

$$\sum_{\phi_{\tilde{\Sigma}},\phi_{\Sigma\pm}} (\Psi_f[\phi_{\Sigma^+}])^* \left(\int_{\phi_{\tilde{\Sigma}}}^{\phi_{\Sigma^+}} [\mathcal{D}\Phi]_{\phi'_L}\, e^{iS_L[\Phi]} \right) \underbrace{\left(\int_{\phi_{\Sigma^-}}^{\phi_{\tilde{\Sigma}}} [\mathcal{D}\Phi]_{\tilde{\phi}}\; e^{i\tilde{S}[\Phi]} \right) \left(\int_0^{\phi_{\Sigma^-}} [\mathcal{D}\Phi]_0\, e^{-S_-[\Phi]} \right)}_{\left(\int_0^{\phi_{\tilde{\Sigma}}} [\mathcal{D}\Phi]_{\{0,\tilde{\phi}\}}\; e^{-S_{\mathcal{M}_-\cup\tilde{\mathcal{M}}}[\Phi]} \right)}$$



Excited States

- The last step is to squeeze the remaining Lorentzian part $\mathcal{M}'_L$ such that

$$\langle\Psi_f|T[e^{-i\int_{\partial_r\mathcal{M}'_L}\mathcal{O}\,\phi'_L}]\left(e^{-\int_{\partial_r\tilde{\mathcal{M}}}\mathcal{O}\,\tilde{\phi}}\,|0\rangle\right) =$$

$$\sum_{\phi_{\tilde{\Sigma}},\phi_{\Sigma+}}(\Psi_f[\phi_{\Sigma+}])^*\left(\int_{\phi_{\tilde{\Sigma}}}^{\phi_{\Sigma+}}[\mathcal{D}\Phi]_{\phi'_L}\,e^{iS_L[\Phi]}\right)\left(\int_0^{\phi_{\tilde{\Sigma}}}[\mathcal{D}\Phi]_{0,\tilde{\phi}}\,e^{-S_{\mathcal{M}_-\cup\tilde{\mathcal{M}}}[\Phi]}\right)$$

becomes

$$\langle\Psi_f|\left(e^{-\int_{\partial_r\tilde{\mathcal{M}}}\mathcal{O}\,\tilde{\phi}}\,|0\rangle\right) = \sum_{\phi_{\Sigma+}}(\Psi_f[\phi_{\Sigma+}])^*\left(\int_0^{\phi_{\Sigma+}}[\mathcal{D}\Phi]_{0,\tilde{\phi}}\,e^{-S_{\mathcal{M}_-\cup\tilde{\mathcal{M}}}[\Phi]}\right)$$

but if $\langle\Psi_f|$ is indeed arbitrary then we have shown that

$$|\Psi^{\phi_-}\rangle \equiv e^{-\int_{\mathcal{S}_-}\mathcal{O}\,\phi_-}|0\rangle$$

Excited States

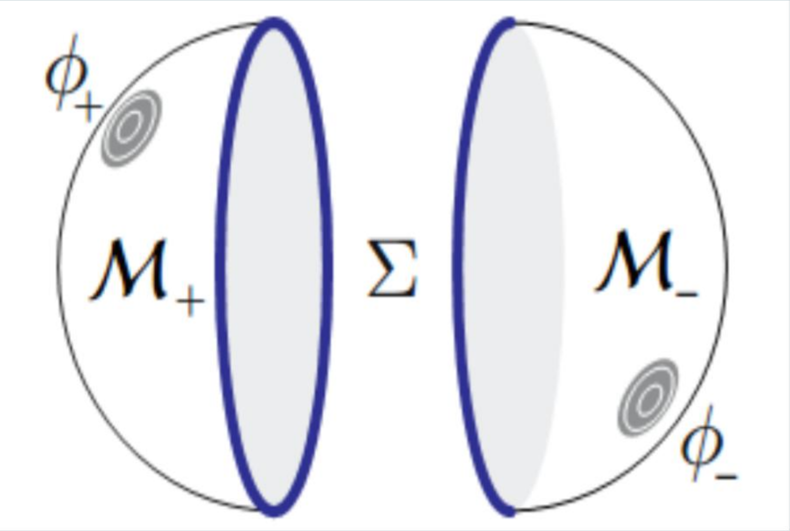
- Having obtained the previous result, one can (out of pure boredom) try to calculate the inner product between them…
- But hold your horses because it turns out that the GKPW prescription has already done that for us!

$$\langle \Psi^{\phi_+} | \Psi^{\phi_-} \rangle = e^{-S_E^0[\phi_+,\phi_-]}$$

$$= e^{-\int_{S^d} dx \int_{S^d} dy\, \phi(x)\, G(x,y)\, \phi(y)}$$

where

$$\phi(x) = \begin{cases} \phi_-(x) \text{ if } x \in \mathcal{M}_- \\ \phi_+(x) \text{ if } x \in \mathcal{M}_+ \end{cases}$$



Excited States

- It is easy to see that $\langle \Psi^{\phi_-} | \Psi^{\phi_-} \rangle \neq 1$. Normalizing the states yields the result

$$\langle \Psi_{\mathcal{N}}^{\phi_+} | \Psi_{\mathcal{N}}^{\phi_-} \rangle = e^{-|\phi_- - \phi_+^{\star}|^2}$$

where the always positive product $(G < 0)$

$$(\phi_1 \, , \, \phi_2) \equiv \int_{-\infty}^{0} d\tau \, \int_{\infty}^{0} d\tau' \, \phi_1(\tau) \, G(\tau, \tau') \, \phi_2^{\star}(\tau')$$

and $\phi^{\star}(\tau, x) \equiv \phi(-\tau, x)$ have been defined.

- Does it ring any bells…? Not yet…?

Excited States

- Let's see if there's some other prescription we can compare to…

Excited States

- The BDHM prescription states that the quantum CFT operator IS the canonically quantized AdS field

$$\hat{\mathcal{O}}(t,\Omega) \equiv \lim_{r\to\infty} r^{\Delta}\,\hat{\Phi}(t,r,\Omega) = \sum_k \hat{a}_k^{\dagger} F_k^{*}(t,\Omega) + \hat{a}_k F_k(t,\Omega)$$

- If both prescription are consistent, then

$$|\Psi_{\mathcal{N}}^{\phi_-}\rangle \propto e^{-\int_{\mathcal{S}_-}\mathcal{O}\,\phi_-}|0\rangle \propto e^{\sum_k \lambda_k^- a_k^{\dagger}}|0\rangle$$

$$\lambda_k^- = -\int_{\partial_r\mathcal{M}_-} d\tau d\Omega\, F_k^{*}(-i\tau,\Omega)\,\phi_-(\tau,\Omega)$$

Excited States

- Before moving on to the concrete example, lets write down some results for these states to see where we are aiming at

$$\langle \Psi_{\mathcal{N}}^{\phi_+} | \Psi_{\mathcal{N}}^{\phi_-} \rangle = e^{-|\phi_- - \phi_+^\star|^2}$$

$$\frac{\langle \Psi^{\phi_+} | \mathcal{O}(t,\Omega) | \Psi^{\phi_-} \rangle}{\langle \Psi^{\phi_+} | \Psi^{\phi_-} \rangle} = \sum_k (\lambda_k^+)^* \ F_k^*(t,\Omega) + \lambda_k^- \ F_k(t,\Omega)$$

$$\left. \frac{\langle \Psi^{\phi_+} | T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')] | \Psi^{\phi_-} \rangle}{\langle \Psi^{\phi_+} | \Psi^{\phi_-} \rangle} \right|_c = C_\Delta \Big(\cos((t-t')(1-i\epsilon)) - \cos(\varphi - \varphi') \Big)^{-\Delta}$$

- Further connected n-point functions are null. Feel free to ask why, since it may not be trivial.

WARNING! SUBTLETIES AHEAD!

- Connected n-point functions in general have multiple terms, take for example n=2

$$\left.\frac{\langle\Psi^{\phi}+|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|\Psi^{\phi}-\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle}\right|_c \equiv \frac{\langle\Psi^{\phi}+|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|\Psi^{\phi}-\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle} - \frac{\langle\Psi^{\phi}+|\mathcal{O}(t,\varphi)|\Psi^{\phi}-\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle}\frac{\langle\Psi^{\phi}+|\mathcal{O}(t',\varphi')|\Psi^{\phi}-\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle}$$

- One can show (won't do it now, don't insist) that for coherent states

$$\left.\frac{\langle\Psi^{\phi}+|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|\Psi^{\phi}-\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle}\right|_c = \frac{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle\langle 0|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|0\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle} = \langle 0|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|0\rangle$$

- Similar cancelations are responsible for n>2 trivial results.

Excited States

- Before moving on to the concrete example, lets write down some results for these states to see where we are aiming at

$$\langle \Psi_{\mathcal{N}}^{\phi_+} | \Psi_{\mathcal{N}}^{\phi_-} \rangle = e^{-|\phi_- - \phi_+^\star|^2}$$

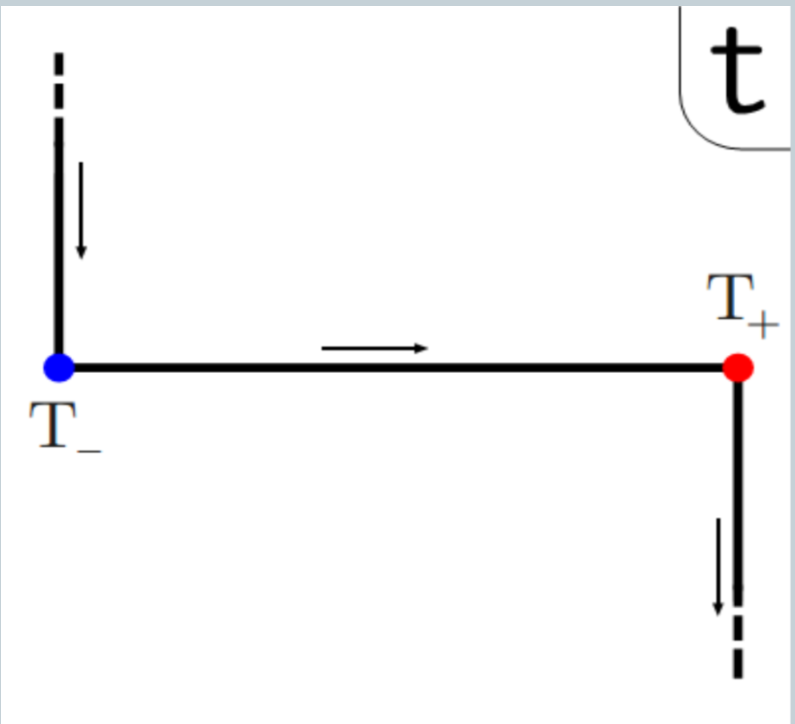
$$\frac{\langle \Psi^{\phi_+} | \mathcal{O}(t,\Omega) | \Psi^{\phi_-} \rangle}{\langle \Psi^{\phi_+} | \Psi^{\phi_-} \rangle} = \sum_k (\lambda_k^+)^* \; F_k^*(t,\Omega) + \lambda_k^- \; F_k(t,\Omega)$$

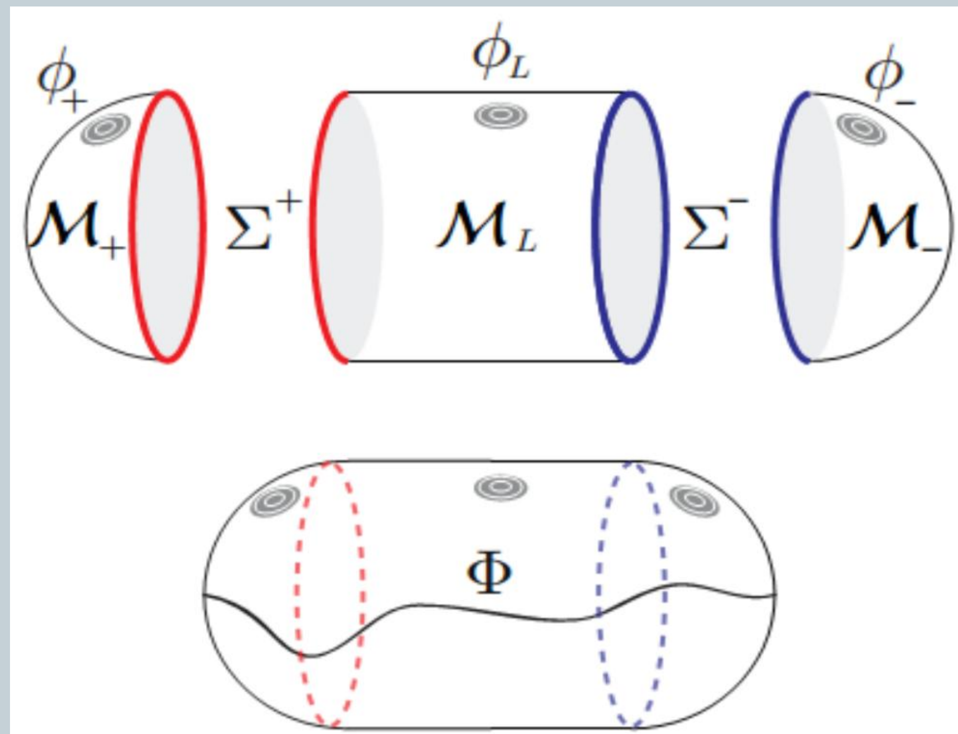
$$\left. \frac{\langle \Psi^{\phi_+} | T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')] | \Psi^{\phi_-} \rangle}{\langle \Psi^{\phi_+} | \Psi^{\phi_-} \rangle} \right|_c = C_\Delta \Big(\cos((t-t')(1-i\epsilon)) - \cos(\varphi - \varphi') \Big)^{-\Delta}$$

- Further connected n-point functions are null. Feel free to ask why, since it may not be trivial.

In-Out Formalism

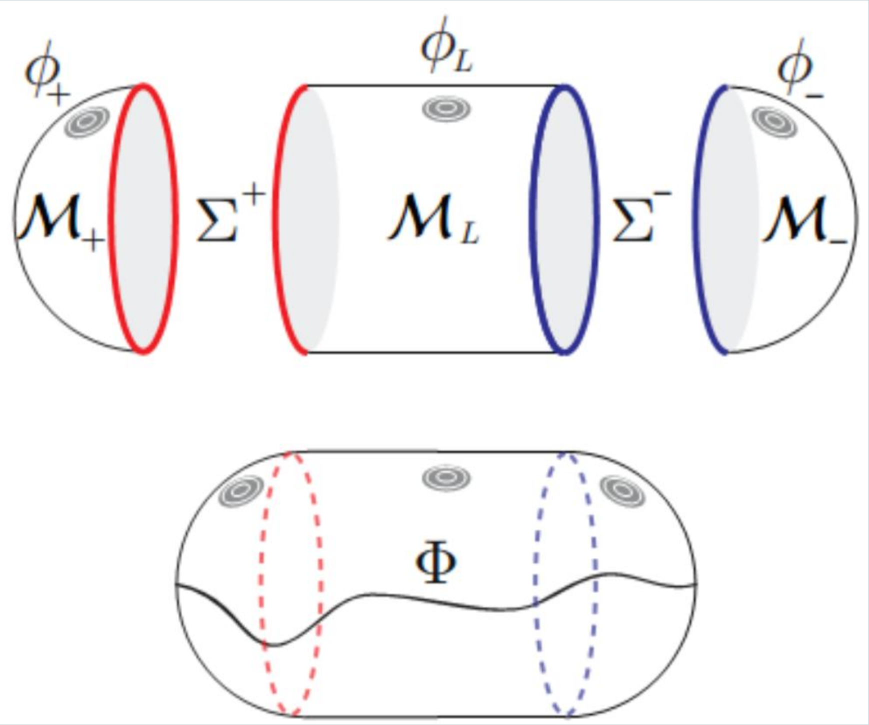
- A massive free complex scalar field dual to a CFT local operator in the In-Out formalism is solved.





In-Out Formalism

- Steps
  - Solve the field EOM
    - Lorentz region
    - Euclidean regions
  - Solve the continuity equations
  - Obtain the on shell action
  - Differentiate!



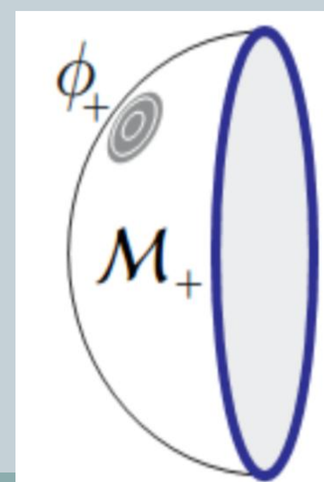
WARNING! SUBTLETIES AHEAD!

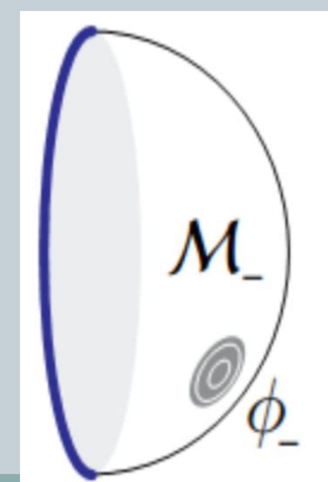
- A radial cut-off R is necessary for the problem to be well defined. For $i = \{\pm, L\}$, the boundary conditions are

$$\Phi^i(r,t,\varphi)|_{r=R} = R^{-\Delta_-}\phi^i(t,\varphi) = R^{\Delta-2}\phi^i(t,\varphi)$$

$$\Delta_+ = \Delta = d/2 + \sqrt{\frac{d^2}{4} + m^2} = 1 + \sqrt{1+m^2}$$

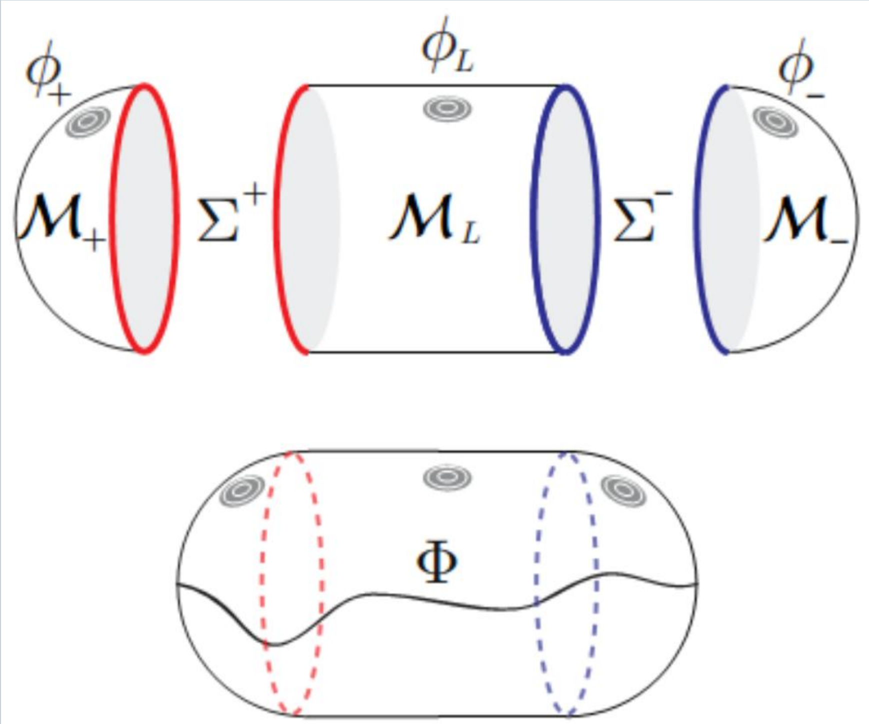
- Our Euclidean manifolds admit normalizable modes!





In-Out Formalism

- Steps
  - Solve the field EOM
    - Lorentz region
    - Euclidean regions
  - Gluing the solutions
  - On-shell action
  - Differentiate!



In-Out Formalism

- Lorentz region
  - Metric and EOM

$$ds^2 = -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 + r^2 d\varphi^2$$

$$\left(\Box - m^2\right)\Phi_L = 0 \implies \Phi_L(r,t,\varphi) \propto e^{-i\omega t + il\varphi} f(\omega,l,r)$$

$$f(\omega,l,r) = (1+r^2)^{\sqrt{\omega^2}/2}\, r^{|l|}\, {}_2F_1\left(\frac{\sqrt{\omega^2}+|l|+\Delta}{2}, \frac{\sqrt{\omega^2}+|l|-\Delta+2}{2}; 1+|l|; -r^2\right)$$

  - For frequencies $\pm\omega_{nl}^R$ one can build N solutions $g_{nl}(r)|_{r=R} = 0$
  - Solution

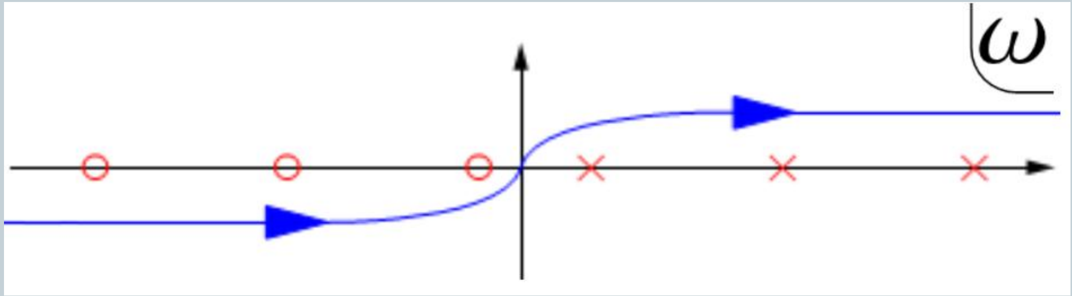
$$\Phi_L(r,t,\varphi) = \frac{R^{\Delta-2}}{4\pi^2}\sum_{l\in\mathbb{Z}}\int_{\mathcal{F}} d\omega dt' d\varphi' e^{-i\omega(t-t')+il(\varphi-\varphi')}\phi_L(t',\varphi')\frac{f(\omega,l,r)}{f(\omega,l,R)} +$$

$$\sum_{\substack{n\in\mathbb{N}\\ l\in\mathbb{Z}}}\left(L_{nl}^{+}\, e^{-i\omega_{nl}^R t} + L_{nl}^{-}\, e^{+i\omega_{nl}^R t}\right)e^{il\varphi}g_{nl}(r)$$

WARNING! SUBTLETIES AHEAD!

- The NN modes integrand as an infinite number of single poles at $\pm\omega_{nl}^{R}$ (this is no coincidence!)

$$\frac{1}{f(\omega,l,R)}$$



- This forces to choose a complex integration path in the frequency integral.

In-Out Formalism

- Lorentz region
  - Metric and EOM

$$ds^2 = -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 + r^2 d\varphi^2$$

$$\left(\Box - m^2\right)\Phi_L = 0 \implies \Phi_L(r,t,\varphi) \propto e^{-i\omega t + il\varphi}\, f(\omega,l,r)$$

$$f(\omega,l,r) = (1+r^2)^{\sqrt{\omega^2}/2}\; r^{|l|}\; {}_2F_1\left(\tfrac{\sqrt{\omega^2}+|l|+\Delta}{2}, \tfrac{\sqrt{\omega^2}+|l|-\Delta+2}{2}; 1+|l|; -r^2\right)$$

  - For frequencies $\pm\omega_{nl}^R$ one can build N solutions $g_{nl}(r)|_{r=R} = 0$
  - Solution

$$\Phi_L(r,t,\varphi) = \tfrac{R^{\Delta-2}}{4\pi^2}\sum_{l\in\mathbb{Z}}\int_{\mathcal{F}} d\omega dt' d\varphi' e^{-i\omega(t-t')+il(\varphi-\varphi')}\phi_L(t',\varphi')\tfrac{f(\omega,l,r)}{f(\omega,l,R)} +$$

$$\sum_{\substack{n\in\mathbb{N}\\ l\in\mathbb{Z}}}\left(L_{nl}^{+}\, e^{-i\omega_{nl}^R t} + L_{nl}^{-}\, e^{+i\omega_{nl}^R t}\right)e^{il\varphi} g_{nl}(r)$$

In-Out Formalism

- Euclidean regions
  - Metric and EOM

$$ds^2 = +(1+r^2)d\tau^2 + (1+r^2)^{-1}dr^2 + r^2 d\varphi^2$$

$$\left(\Box - m^2\right)\Phi_\pm = 0 \implies \Phi_\pm(r,\tau,\varphi) \propto e^{i\omega\tau + il\varphi} f(-i\omega,l,r)$$

$$f(\omega,l,r) = (1+r^2)^{\sqrt{\omega^2}/2}\, r^{|l|}\, {}_2F_1\left(\tfrac{\sqrt{\omega^2}+|l|+\Delta}{2}, \tfrac{\sqrt{\omega^2}+|l|-\Delta+2}{2}; 1+|l|; -r^2\right)$$

  - For frequencies $\pm i\omega_{nl}^R$ one can still build N solutions!
  - Solution for $\mathcal{M}_+$

$$\Phi_+(r,\tau,\varphi) = \tfrac{R^{\Delta-2}}{4\pi^2} \textstyle\sum_{l\in\mathbb{Z}} \int d\omega d\tau' d\varphi' e^{i\omega(\tau-\tau')+il(\varphi-\varphi')} \phi_+(\tau',\varphi') \tfrac{f(-i\omega,l,r)}{f(-i\omega,l,R)} +$$

$$\sum_{\substack{n\in\mathbb{N}\\ l\in\mathbb{Z}}} E_{nl}^+\, e^{-\omega_{nl}^R(\tau+iT)+il\varphi} g_{nl}(r)$$

In-Out Formalism

- Gluing the solutions
  - Near $\sum^+$, $(t-t')>0$ since every source $\phi^L(t')$ has been left behind.
  - One can then carry out the ω integrals in the NN modes by residue theorem

$$\frac{R^{\Delta-2}}{4\pi^2}\sum_{l\in\mathbb{Z}}\int_{\mathcal{F}} d\omega dt' d\varphi' e^{-i\omega(t-t')+il(\varphi-\varphi')}\phi_L(t',\varphi')\frac{f(\omega,l,r)}{f(\omega,l,R)} =$$

$$iR^{\Delta-2}\sum_{nl} Res^R_{nl}\phi^*_{L;nl}e^{-i\omega^R_{nl}t+il\varphi}g_{nl}(r)$$

  - Thus the Lorentzian solution near Σ^+ can be written

$$\Phi_L(r,t,\varphi)\sim\sum_{nl}\left(\left(L^+_{nl}+iR^{\Delta-2}Res^R_{nl}\phi^*_{L;nl}\right)e^{-i\omega^R_{nl}t}+L^-_{nl}e^{i\omega^R_{nl}t}\right)e^{il\varphi}g_{nl}(r)$$

In-Out Formalism

- Gluing the solutions
  - Carrying out a similar process for each solution, one finds that the continuity equations are satisfied if

$$L_{nl}^{\pm} = R^{\Delta-2} Res_{nl}^{R}\, \phi_{\mp;n(-l)}$$

$$E_{nl}^{+} = R^{\Delta-2} Res_{nl}^{R} \left(i\phi_{L;nl}^{*} + \phi_{-;n(-l)} \right)$$

$$E_{nl}^{-} = R^{\Delta-2} Res_{nl}^{R} \left(i\phi_{L;n(-l)} + \phi_{+;n(-l)} \right)$$

In-Out Formalism

- On-shell action

$$S^0 = -\frac{1}{2}\lim_{R\to\infty}\left[\sum_{i=\pm,L}\int_{\partial_r\mathcal{M}_i} dt_i\, d\varphi\,(1+r^2)\,(R^{\Delta-2}\phi_i)\,r\partial_r\Phi\right]_{r=R}$$

  - The radial derivative applied to NN modes give

$$R^{\Delta-2}\frac{r\partial_r f(\omega,l,r)}{f(\omega,l,R)}\bigg|_{r=R} \sim \mathbb{S}[l,R] - 2(\Delta-1)R^{-\Delta}\frac{B(\omega,l)}{A(\omega,l)}(1+o(R^{-2}))$$

while for N modes

$$L^{\pm}_{nl}\,r\partial_r g_{nl}(r)\Big|_{r=R} \sim -2(\Delta-1)R^{-\Delta}\,B_{nl}\,Res_{nl}\phi_{\mp;n(-l)}(1+o(R^{-2}))$$

WARNING! SUBTLETIES AHEAD!

- You may wonder how come both N and NN modes end up with the same R dependence!
  - There are 2 key arguments that result in such a blasphemous result:
    - We are not considering the r dependence, but rather the R dependence! The solutions are of the form

$$\Phi_{NN} \sim A(R) f(\omega, l, r) \qquad \Phi_N \sim B(R) g_{nl}(r)$$

As a consequence, it is not immediate to determine the final R dependence in a large-R expansion.

    - Both the $r\partial_r$ operator and the denominator $f(\omega, l, R)$ in the NN solution play a subtle but fundamental role in the process! These result in turning the "real" NN leading terms in contact terms.

WARNING! SUBTLETIES AHEAD!

- You may wonder how come both N and NN modes end up with the same R dependence!
  - For NN modes, the $R^{-\Delta}$ term is the leading one that has non-integer dependence in l, i.e. that is not a contact term.
  - For N modes, it can be shown that

$$g_{nl}(r) \sim -B(\omega_{nl}^R, l)\, R^{-\Delta} \left(\left(\tfrac{r}{R}\right)^{\Delta-2} - \left(\tfrac{r}{R}\right)^{-\Delta} \right)$$

and that the coefficients are independent or R

$$\lim_{R\to\infty} L_{nl}^{\pm} \propto \lim_{R\to\infty} R^{\Delta-2} Res_{nl}^{R} = \frac{1}{2\pi i} \oint_{\omega=-\omega_{nl}} d\omega \frac{1}{A(\omega,l)}$$

which allows to prove the previous results.

In-Out Formalism

- On-shell action

$$S^0 = -\tfrac{1}{2}\lim_{R\to\infty}\left[\sum_{i=\pm,L}\int_{\partial_r\mathcal{M}_i} dt_i\, d\varphi\,(1+r^2)\,(R^{\Delta-2}\phi_i)\,r\partial_r\Phi\right]_{r=R}$$

  - The radial derivative applied to NN modes give

$$R^{\Delta-2}\frac{r\partial_r f(\omega,l,r)}{f(\omega,l,R)}\bigg|_{r=R} \sim \mathbb{S}[l,R] - 2(\Delta-1)R^{-\Delta}\frac{B(\omega,l)}{A(\omega,l)}(1+o(R^{-2}))$$

while for N modes

$$L^{\pm}_{nl}\, r\partial_r g_{nl}(r)\Big|_{r=R} \sim -2(\Delta-1)R^{-\Delta}\, B_{nl}\, Res_{nl}\phi_{\mp;n(-l)}(1+o(R^{-2}))$$

In-Out Formalism

- On-shell action
  - The same analysis holds for each $\mathcal{M}_i$ and putting together the three contributions one gets the full on-shell action.
  - Its complete expression is not very illuminating, but noticing that each term in the action is $\sim \phi^i r\partial_r(\phi^i + \sum_{j\neq i}\phi^j)$:
    - There are terms independent of ϕ^L coming from the Euclidean quadratic terms.
    - There are terms linear in ϕ^L coming from the mixed $\phi^i\phi^j$ terms.
    - There is one term quadratic in ϕ^L .
    - No other powers of ϕ^L appear in the action.
  - Notice that this means that that the connected n-point functions, n>2 are all trivial in this example. (COHERENCE!)

In-Out Formalism

- Inner Product

$$\ln\langle\Psi^{\phi_+}|\Psi^{\phi_-}\rangle = \frac{1}{2}\int d\tau\, d\varphi\, d\tau'\, d\varphi'\, \phi(\tau,\varphi)\mathcal{P}(\tau,\tau',\varphi,\varphi')\phi(\tau',\varphi')$$

where

$$\phi(\tau,\varphi) = \begin{cases} \phi_-(\tau,\varphi) \text{ if } \tau\in\mathcal{M}_- \\ \phi_+(\tau,\varphi) \text{ if } \tau\in\mathcal{M}_+ \end{cases}$$

$$\mathcal{P}(\tau,\tau',\varphi,\varphi') = -2G(\tau,\tau',\varphi,\varphi') = \frac{(\Delta-1)^2}{2^{\Delta-1}\pi}\Big(\cosh(\tau-\tau') - \cos(\varphi-\varphi')\Big)^{-\Delta}$$

In-Out Formalism

- One Point function

$$\omega_{nl} = 2n + |l| + \Delta$$

$$\frac{\langle \Psi^{\phi_+} | \mathcal{O}(t,\varphi) | \Psi^{\phi_-} \rangle}{\langle \Psi^{\phi_+} | \Psi^{\phi_-} \rangle} = -2(\Delta-1) \sum_{nl} B_{nl} Res_{nl} \left(\phi_{+;nl}\ e^{i\omega_{nl}t - il\varphi} + \phi_{-;n(-l)}\ e^{-i\omega_{nl}t + il\varphi} \right)$$

which (up to a normalization constant) perfectly matches the BHDM results, behaving as if

$$|\Psi^{\phi_-}_{\mathcal{N}}\rangle \propto e^{-\int_{\mathcal{S}_-} \mathcal{O}\,\phi_-} |0\rangle \propto e^{\sum_k \lambda_k^- a_k^\dagger} |0\rangle$$

$$\lambda_{nl}^- = -\sqrt{2\pi}\sqrt{2(\Delta-1)B_{nl}Res_{nl}}\,\phi_{-;nl} = -C_\Delta \int_{\partial_r \mathcal{M}_-} d\tau d\varphi\, F^*_{nl}(-i\tau,\varphi)\,\phi_-(\tau,\varphi)$$

In-Out Formalism

- Two Point function

$$\left.\frac{\langle\Psi^{\phi}+|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|\Psi^{\phi}-\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle}\right|_c = \frac{(\Delta-1)^2}{2^{\Delta-1}\pi}\Big(\cos((t-t')(1-i\epsilon))-\cos(\varphi-\varphi')\Big)^{-\Delta}$$

which is the same as for the vacuum… Wait, WHAT!?

- Recall this is the connected 2-point function!

$$\left.\frac{\langle\Psi^{\phi}+|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|\Psi^{\phi}-\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle}\right|_c = \frac{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle\langle 0|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|0\rangle}{\langle\Psi^{\phi}+|\Psi^{\phi}-\rangle} = \langle 0|T[\mathcal{O}(t,\varphi)\mathcal{O}(t',\varphi')]|0\rangle$$

Conclusions

- SvR recipe is recovered from quantum analysis and "link" with EQG.
- The states generated by turning on Euclidean sources are

$$|\Psi^{\phi_-}\rangle = e^{-\int_{\mathcal{S}_-} \mathcal{O}\,\phi_-}|0\rangle$$

it proves the SvR conjecture on excited states.

- These states reproduce every holographic check from coherent states, with explicit eigenvalues

$$\lambda^{\pm}_{nl} = -\sqrt{2\pi}\sqrt{2(\Delta-1)B_{nl}Res_{nl}}\,\phi_{\pm;n(\mp l)}$$

Current Objectives

- Introduce $\lambda\Phi^3$ interactions
  - We expect that a Φ^3 term lead an on-shell action with contributions as $\Phi^2 \sim (a^\dagger + a)^2$ which could translate into a sort of squeezed states.
- Prescribe our own operator
  - Coherent states are an (over-)complete basis, and as such allows us to reconstruct the CFT operator from the bulk theory.
- Improvement of the Hartle-Hawking construction
  - For AdS Quantum Gravity we know how to define/compute wave functional of excited states through Euclidean path integrals.