ELECTROMAGNETIC SHIFT IN THE EULER-HEISENBERG THEORY AND RADIATION FOR A BORN-INFELD ACCELERATED CHARGE

Bonetti Luca

Supervisor: Alessandro Spallicci

Collaborators: Santiago E. Perez Bergliaffa[1]
José Abdalla Halayël-Nieto[2]

[1]Departamento de Fisica Teorica, UERJ, Brazil
[2]Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro, Brazil

OUTLINE

- Introduction
- Non Linear theories
  - Classical non linearities
  - Quantum non linearities
- Euler-Heisenberg Dipole
  - Effective metric formalism
  - Electromagnetic shift
  - Results
- Accelerating particle in Born-Infeld
  - Problems due to non linearities
  - Rindler’s spacetime
  - Our approach to the problem
- Conclusions

INTRODUCTION

- The main theory explaining electromagnetic phenomena is due to Maxwell who summarized a couple centuries of experiments into his equations

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho/\epsilon_0 && \textit{Gauss' law (electric)} \\
\nabla \cdot \mathbf{B} &= 0 && \textit{Gauss' law (magnetic)} \\
\nabla \times \mathbf{E} &= -\partial_t \mathbf{B} && \textit{Faraday's law} \\
\nabla \times \mathbf{B} &= \mu_0(\mathbf{j} + \epsilon_0\, \partial_t\, \mathbf{E}) && \textit{Ampère-Maxwell law}
\end{aligned}
$$

- This theory is linear

- Its Lagrangian can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$
\begin{aligned}
A^\mu &= \left(\phi, \vec{A}\right) \\
\vec{\nabla} \times \vec{A} &= \vec{B} \\
-\vec{\nabla}\phi &= \vec{E}
\end{aligned}
$$

A_μ being the 4-potential

INTRODUCTION

- From a quantum perspective, to A_μ it is associated a particle, the photon.
- There is no mass term in the Lagrangian, hence Maxwell's theory implies a massless photon.
- Maxwell's theory works very well, as proven in the last couple centuries. But it has some limitation too.

In fact:

- The masslessness of the photon cannot be proven. Only a lower experimental limit can be given.
- The superposition principle can also be proven up to a certain precision.
- Adapting Maxwell's theory to the quantum framework is not straightforward and somewhat artificial.

NON LINEAR THEORIES

A lot can be said regarding non-Maxwellian theories. As we have pointed out, the two main groups of theories that go beyond the standard one are those who predict a massive photon and those who are non linear.

We will focus on these latter.

The non linearities can be divided into two classes:

- CLASSICAL NON LINEARITIES: when also at classical level we have non linear terms in the Lagrangian

- QUANTUM NON LINEARITIES: when the non linearities are introduced to explain the non linear behavior of QED beyond tree level.

NON LINEAR THEORIES

CLASSICAL NON LINEARITY

- On the macroscopic level the superposition principle is widely proven with a 0,1% precision (EM field generated by a group of charges and currents, transformers, standing waves, diffraction patterns in optic, X-ray crystallography, refraction…)

- Problems might arise at atomic or nuclear level. If we think of a charged particles as a localized distribution of charge, then it is obvious from Maxwell's theory that the field strength and, hence, the electromagnetic energy increase as the charge is more and more localized.

- For atoms, linearity has been tested up to 10^9-10^{15} V/cm fields, the ones one can find in electron orbits. Energy level distances in light atoms allow us to say that the experimental data are in agreement with superposition up to 1 part in 10^6.

- In nuclei field strength of the order of 10^{19} V/cm can be reached. In this regime Coulomb energies of heavy nuclei have been tested and found in agreement with linearity up to 1 part in 10^6.

NON LINEAR THEORIES

BORN INFELD THEORY

- Despite these constraining limits, efforts have been done to create a non-linear classical theory. The main reason behind this effort is to avoid infinities.

$$U = \frac{1}{2} \int d^3x \int d^3x' \frac{\rho(x)\rho(x')}{|\vec{x}-\vec{x}'|}$$

- The first and most relevant try was that of **Born and Infeld** in 1934*.

There are 3 fundamental points that bring Born and Infeld to their theory:

- **Principle of finiteness**: a physical theory must imply only finite measurable quantities
- **Parallel to relativity**: the Lagrangian is deduced mimicking what happens in relativity where the kinetic energy must take into account the light speed upper limit for velocities.
- **Unitary principle**: it exists only one physical entity, the electromagnetic field; matter particles are considered singularities in the field and mass is a derived notion expressed in term of electromagnetic energy (**electromagnetic mass**)

*Born M., Infeld L., 1934, Proc. Roy. Soc. London A, 144, 425

NON LINEAR THEORIES

BORN INFELD THEORY

$$\frac{1}{2}mv^2 \quad \Longrightarrow \quad mc^2\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\mathcal{L}_M = F \quad \Longrightarrow \quad \mathcal{L}_{BI} = 1-\sqrt{1+2\frac{F}{b^2}-\frac{G^2}{b^4}}$$

$$F = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad \wedge \qquad G = \frac{1}{4}F_{\mu\nu}\mathcal{F}^{\mu\nu}$$

$$\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} \qquad \epsilon^{\mu\nu\lambda\rho} = \begin{cases} 1 & if\ \mu\nu\lambda\rho\ is\ an\ even\ permutation\ of\ 0123 \\ -1 & if\ \mu\nu\lambda\rho\ is\ an\ odd\ permutation\ of\ 0123 \\ 0 & otherwise \end{cases}$$

NON LINEAR THEORIES

BORN INFELD THEORY

- Keeping in mind that

$$F = \frac{1}{4}\left(B^2 - E^2\right) \qquad \wedge \qquad G = E \cdot B$$

one sees that the term G^2 is of the fourth order in the fields, whereas F is only quadratic. Hence it is negligible unless we are extremely close to the field source.
Therefore, the BI Lagrangian can be simplified into

$$\mathcal{L}_{BI} = 1 - \sqrt{1 + 2\frac{F}{b^2}}$$

For this Lagrangian the equations of motion are

$$\partial_\mu \left(F^{\mu\nu} \left(1 + 2F\right)^{-\frac{1}{2}} \right) = j^\nu \qquad \wedge \qquad \partial_\mu \mathcal{F}^{\mu\nu} = 0$$

NON LINEAR THEORIES

BORN INFELD THEORY

- In terms of the fields they can be written as

$$\begin{aligned} \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \qquad\qquad \vec{\nabla} \cdot \vec{D} = \rho \\ \vec{\nabla} \times \vec{H} - \frac{\partial \vec{E}}{\partial t} &= \vec{j} \qquad\qquad \vec{\nabla} \cdot \vec{B} = 0 \end{aligned}$$

where

$$\vec{D} = \frac{\vec{E}}{\sqrt{1 - \frac{B^2 - E^2}{2b^2}}} \qquad\qquad \vec{H} = \frac{\vec{B}}{\sqrt{1 - \frac{B^2 - E^2}{2b^2}}}$$

One sees that these equations are Maxwell's ones, just written for the field D and H. Therefore studying BI electrodynamics is equivalent to studying Maxwell's one in a non linear medium whose magnetic susceptibility and dielectric constant are

$$\mu = \epsilon = \frac{1}{\sqrt{1 - \frac{B^2 - E^2}{2b^2}}}$$

NON LINEAR THEORIES

BORN INFELD THEORY

- The physical field E is non singular at the origin. It’s the unphysical D field that is singular.

Electrostatic, time independent ➡ $\mathrm{div} D = 0$

Spherical symmetry ➡ $$\frac{d}{dr}\left(r D_r\right) = 0 \Rightarrow D_r = \frac{e}{r^2}$$

$$\Rightarrow D_r = -\frac{\phi_r'}{\sqrt{1-\frac{\phi'^2}{b?2}}} \qquad \left(E_r = -\phi_r'\right)$$

⬅ $$\phi\left(r\right) = \frac{e}{r_0} f\left(\frac{r}{r_0}\right)$$

$$r_0 = \sqrt{\frac{e}{b}}$$

$$f\left(x\right) = \int_x^{\infty} \frac{dy}{\sqrt{1+y^4}}$$

$$x = \tan\tfrac{1}{2}\beta \qquad \bar{\beta} = 2\arctan x$$

$$f\left(x\right) = \frac{1}{2}\int_{\bar{\beta}(x)}^{\pi} \frac{d\beta}{\sqrt{1-\frac{1}{2}\sin^2\beta}} = \frac{1}{2} F\left(\frac{1}{\sqrt{2}}, \bar{\beta}\right)$$

NON LINEAR THEORIES

So

$$f(0) = \frac{1}{2} F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) = 1.8541 \Rightarrow \phi(0) = 1.8541 \frac{e}{r_0}$$

Other interesting features of Born-Infeld Theory are:

- The electron is seen as the discontinuity in the E field and its self energy is finite
- It's the only non linear theory yielding no birefringence
- It predicts metastable orbits for scattering from a Coulomb potential
- It is derivable through string theory for between oriented open and closed strings in D=4

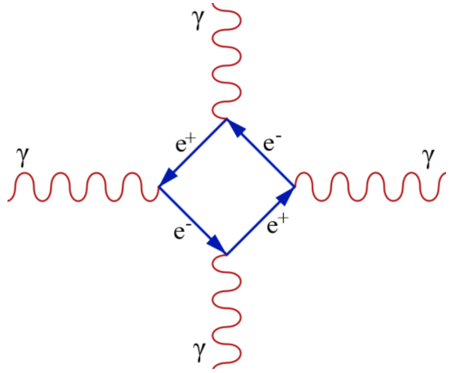
NON LINEAR THEORIES

QUANTUM NON LINEARITY

- Unlike what happens on the classical level, we have several evidences of the deviation from linearity in the quantum regime. This is due to the uncertainty principle which allows the momentary creation of electron-positron couple and subsequent annihilation with creation of two photons.

- This phenomenon is known as light-light scattering.

- When two plane waves defined by the wave vectors k_1 and k_2 scatter, on top of the leading linear term one has a small probability of them interacting like shown in the diagram, transforming into two different waves with vectors k_3 and k_4



NON LINEAR THEORIES

QUANTUM NON LINEARITY

- Similarly to what happens with BI theory, even the effects of light light scattering can be reproduced – for slow varying fields – by electric and magnetic permeability tensors

$$D_i = \sum_k \epsilon_{ik} E_k \qquad\qquad H_i = \sum_k \mu_{ik} B_k$$

with* – in the low frequency fields (for fast varying fields see **) –

$$\epsilon_{ik} = \delta_{ik} + \frac{e^4 \hbar}{45 \pi m^4 c^7} \left[2 \left(E^2 - B^2 \right) \delta_{ik} + 7 B_i B_k \right] + \ldots$$
$$\mu_{ik} = \delta_{ik} + \frac{e^4 \hbar}{45 \pi m^4 c^7} \left[2 \left(E^2 - B^2 \right) \delta_{ik} + 7 B E_i E_k \right] + \ldots$$

*: Euler H., Kockel B., 1935, Naturwiss., 23, 246
**: Akhieser A., Landau L., Pomeranchuck I., 1936, Nature, 138, 206

NON LINEAR THEORIES

EULER-HEISENBERG THEORY

- It is a generalization of what we have presented so far.

- Euler and Kockel derived the leading non linear correction due to vacuum polarization in a constant background field. With their Lagrangian Euler and Heisenberg were able to give a close form for all perturbative orders.

$$\mathcal{L} = \underbrace{\frac{e^2}{hc}}_{\alpha}\int_0^\infty \frac{d\eta}{\eta^3}e^{-\eta}\left\{ i\eta^2(\vec{E}.\vec{B})\frac{\left[\cos\left(\frac{\eta}{\mathcal{E}_c}\sqrt{\vec{E}^2-\vec{B}^2+2i(\vec{E}.\vec{B})}\right)+\text{c.c.}\right]}{\left[\cos\left(\frac{\eta}{\mathcal{E}_c}\sqrt{\vec{E}^2-\vec{B}^2+2i(\vec{E}.\vec{B})}\right)-\text{c.c.}\right]}+\mathcal{E}_c^2+\frac{\eta^2}{3}(\vec{B}^2-\vec{E}^2)\right\}$$

$$\mathcal{E}_c = \frac{m^2c^3}{e\hbar} \approx 10^{16}\mathbf{V}/\text{cm}$$

It’s called **critical field**. It is the field strength at which Dirac sea electrons are expected to tunnel into the continuum and produce electron-positron pairs. In modern terms it’s the Schwinger critical field that sets the threshold for nonlinear QED to become relevant.

NON LINEAR THEORIES

EULER-HEISENBERG THEORY

- To the Schwinger critical field corresponds a magnetic critical field, which is the field strength needed for the associated energy of a cyclotron pulsation for an electron to be equal to its rest mass.

$$B_c \approx 4.4 \times 10^{13}\ G$$

- Expanding EH Lagrangian to the quartic order in the perturbation field one obtains Euler and Kockel's results.

- Through their Lagrangian, Euler and Heisenberg were able to model the quantum vacuum instability. Not only, picturing the pair creation process as the tunneling from the Dirac sea into the continuum, they were able to compute the correct rate for the process.

$$\exp\left[-m^2c^3\pi/(\hbar e|E|)\right]$$

NON LINEAR THEORIES

EULER-HEISENBERG THEORY

- This Lagrangian is valid in the case of a constant background field. In any case it has been pointed out by Dunne* that it can be used also in the case of inhomogeneous fields through a derivative expansion.

- Like Euler and Kockel's result also EH Lagrangian has been computed for low frequencies, the so called soft photon regime $\omega \ll m_e$.

- The result can be rewritten as

$$\mathcal{L} = -F - \frac{\alpha}{8\pi^2} \int_0^{i\infty} \frac{ds}{s^3} e^{-m^2 s} \left[(es)^2 ab \coth(eas) \cot(ebs) - \frac{(es)^2}{3}(a^2 - b^2) - 1 \right]$$

$$a = \left[(F^2 + G^2)^{1/2} + F \right]^{1/2} \qquad b = \left[(F^2 + G^2)^{1/2} - F \right]^{1/2}$$

*: Dunne G.V., in *From fields to strings: circumnavigating theoretical physics, Ian Kogan memorial collection*, edited by Shifman M,m Vainshtein A. and Wheater J. (World Scientific, Singapore, 2004), vol 1, p. 445

EULER-HEISENBERG DIPOLE

EULER-HEISENBERG THEORY

- Euler-Heisenberg theory is being used right now in PVLAS and BMV (Toulouse) experiment, which aims to show that laser beams can be deviated by a more concentrated pulse. According to EH prediction, this effect should be visible starting from 100 petawatt.

- It predicts birefringence

- Due to the different dispersion relation for the two photon polarization it allows photon splitting.

- The theory has been generalized for non Abelian fields, contributing to the study of gluon fields.

- It can be applied in the context of string theory. If written in the proper time, it’s a useful too to study gauge theories in curved space time and to investigate particle production on cosmological scale.

EULER-HEISENBERG DIPOLE

- Deviations from Maxwell's theory have been sought in terrestrial laboratories for a long time. Among these many Euler-Heisenberg theory predictions like the Schwinger effect, vacuum birefringence under a strong magnetic field or photon splitting in lasers.

- One might try to see if there are any astrophysical phenomena where the same deviations may be relevant. Magnetars are stars endowed with an overcritical field (estimated 10-10^2 B_C at the surface); therefore they are apt to be used as a test for Euler-Heisenberg theory.

- Many consequences have already been studied: light lensing due to the optical property of the vacuum in presence of a magnetic field(1), polarization phase lag(2), how quantum vacuum friction influences pulsar spindown(3).

- We want to investigate how the nonlinearity of the theory affects the photon propagation, calculating the red- or blueshift due to this. As a first step toward computing the effect for a real magnetar, we investigate the magnitude of the EM shift felt from a photon that is emitted from an EH dipole endowed with overcritical field.

(1) : A. Dupays, C. Robilliard, C. Rizzo, and G. Bignami, Phys. Rev. Lett. 96, 161101 (2005)
(2) : J. S. Heyl and N. J. Shaviv, Mon. Not. R. Astr. Soc. 311, 555 (2000)
(3) : A. Dupays, C. Rizzo, and G. Bignami, Europhys. Lett. 98, 49001 (2012)

EULER-HEISENBERG DIPOLE

EFFECTIVE METRIC

- The consequences of non linearity on photon propagation are easily seen in the effective metric formalism

- High energy excitations of a Non Linear (NL) electromagnetic theory on a fixed background propagate in an effective metric that is written in terms of the **background metric**, the **background EM field configuration** and on the details of the **non linearity of the theory**.

- Let's consider a characteristic surface for wave propagation Σ. Following Hadamard* one assumes that the field itself is continuous on Σ whereas its derivative is not

$$[F_{\mu\nu}]_\Sigma = 0, \qquad [J]_\Sigma \equiv \lim_{\delta \to 0^+} \left(J|_{\Sigma+\delta} - J|_{\Sigma-\delta} \right) \qquad k_\lambda = \partial_\lambda \Sigma$$

$$[\partial_\lambda F_{\mu\nu}]_\Sigma = f_{\mu\nu} k_\lambda$$

$f_{\mu\nu} \Rightarrow$ Discontinuity tensor

*: J. Hadamard, «Leçons Sur la Propagation des Ondes et les Equations de l'Hydrodynamique », Hermann, Paris, 1903

EULER-HEISENBERG DIPOLE

Let's take a general Lagrangian depending on the invariants F and G, defined above

$$L = L(F, G)$$

The equations of motion are

$$\partial_\nu (L_F F^{\mu\nu} + L_G \overset{*}{F}{}^{\mu\nu}) = 0.$$

Where

$$\overset{*}{F}{}^{\nu\nu} = \mathcal{F}^{\mu\nu} \qquad \mathcal{L}_X \equiv \partial\mathcal{L}/\partial X$$

Applying the discontinuity the dynamics is described by

$$f_{\mu\nu} k_\lambda + f_{\nu\lambda} k_\mu + f_{\lambda\mu} k_\nu = 0 \quad \Longrightarrow \quad g_\pm^{\mu\nu} k_\mu k_\nu = 0,$$

EULER-HEISENBERG DIPOLE

With some algebra one sees that he expression for g is given by the solution to a second order equation for an auxiliary variable **Ω**

$$\Omega^2\Omega_1+\Omega\Omega_2+\Omega_3=0,$$

$$\Omega_1=-L_F L_{FG}+2FL_{FG}L_{GG}+G(L_{GG}^2-L_{FG}^2),$$

$$\Omega_2=(L_F+2GL_{FG})(L_{GG}-L_{FF})+2F(L_{FF}L_{GG}$$
$$+L_{FG}^2),$$

$$\Omega_3=L_F L_{FG}+2FL_{FF}L_{FG}+G(L_{FG}^2-L_{FF}^2).$$

$$\Omega_{\pm}=\frac{-\Omega_2\pm\sqrt{\Delta}}{2\Omega_1},$$

$$\Delta\doteq(\Omega_2)^2-4\Omega_1\Omega_3$$

So the two effective metrics are given by

$$g_{\pm}^{\mu\nu}=L_F\eta^{\mu\nu}-4[(L_{FF}+\Omega_{\pm}L_{FG})F^{\mu}_{\lambda}F^{\lambda\mu}+(L_{FG}$$

$$+\Omega_{\pm}L_{GG})F^{\mu}_{\lambda}\overset{*}{F}{}^{\lambda\nu}].$$

EULER-HEISENBERG DIPOLE

EFFECTIVE METRIC

- Keeping only terms linear in the perturbations and applying the eikonal approximation, which allows to reduce the differential equations to a single variable, in the high energy limit the two metrics can be written as

$$^{(1)}\tilde{g}^{\mu\nu} = \left[\mathcal{L}_F\, \eta^{\mu\nu} - 4\,\mathcal{L}_{FF}\, F^{\mu}{}_{\alpha}\, F^{\alpha\nu}\right]_0 .$$

$$^{(2)}\tilde{g}^{\mu\nu} = \left[\left(\mathcal{L}_F - 2F\mathcal{L}_{GG}\right)\, \eta^{\mu\nu} - 4\,\mathcal{L}_{GG}\, F^{\mu}{}_{\alpha}\, F^{\alpha\nu}\right]_0$$

where the index 0 means that the quantities are evaluated at the background field.

EULER-HEISENBERG DIPOLE

ELECTROMAGNETIC SHIFT FOR AN EULER-HEISENBERG PHOTON

- The calculation of the redshift for non linear theories must be taken with care, because, in such theories, the photon is accelerated by the non linearities. Therefore it doesn't move on background geodesics

- The motion is governed by

$$p^\mu \widetilde{\nabla}_\mu p^\nu = 0$$

where the covariant derivative is constructed – for each polarization – with the effective metric.

- Defining

$$\widetilde{p}_K \equiv \widetilde{g}^{\mu\nu} p_\mu K_\nu$$

Where $K\mu$ is a Killing vector of $g^{\mu\nu}$ (and hence of $\tilde{g}^{\mu\nu}$), one has

$$\frac{d\widetilde{p}_K}{d\lambda} = 0$$

EULER-HEISENBERG DIPOLE

ELECTROMAGNETIC SHIFT FOR AN EULER-HEISENBERG PHOTON

$$K_\mu = \delta^0_\mu \quad \Rightarrow \quad \widetilde{p}_K = \widetilde{g}^{00} p_0$$ Conserved quantity

$$\Rightarrow \quad p_0 = (\widetilde{g}^{00})^{-1} \widetilde{p}_K$$

- From the definition of redshift

$$1 + z = \frac{g^{00}(E) u_0(E) p_0(E)}{g^{00}(R) u_0(R) p_0(R)}$$

- Using $u_0 = \frac{1}{\sqrt{g^{00}}}$

$$1 + z = \sqrt{\frac{g_{00}(R)}{g_{00}(E)}} \frac{p_0(E)}{p_0(R)}$$

one has

$$= \sqrt{\frac{g_{00}(R)}{g_{00}(E)}} \frac{(\widetilde{g}^{00})^{-1}(E)}{(\widetilde{g}^{00})^{-1}(R)}$$

EULER-HEISENBERG DIPOLE

ELECTROMAGNETIC SHIFT FOR AN EULER-HEISENBERG PHOTON

- The shift can be obviously expressed in terms of the 00 component of the metric as

$$1+z=\frac{(\tilde{g}^{00})^{-1}(E)}{(\tilde{g}^{00})^{-1}(R)}$$

where r_E is the emission point and r_R is the reception point, conveniently taken at infinity where the metric reduces to Minkowski's, the flat one. This formula corrects the results given in some previous papers, where a square root was added to the right hand side.

- In view of the previous results

$$^{(1)}z=\mathcal{L}_F\left(r_E\right)-1$$

$$^{(2)}z=\left[\mathcal{L}_F\left(r_E\right)-2F\left(r_E\right)\mathcal{L}_{GG}\left(r_E\right)\right]-1$$

EULER-HEISENBERG DIPOLE

- We specialize now to the Euler-Heisenberg Lagrangian in the case of zero electric field

$$F = B^2/2,\; G = 0$$

- The derivatives of EH Lagrangian read*

$$\begin{aligned}\mathcal{L}_F &= -1 - \frac{\alpha}{2\pi}\left[\frac{1}{3} + \frac{1}{2\xi^2} - 8\zeta'\left(-1, \frac{1}{2\xi}\right) + \frac{1}{2\xi}\ln\Gamma\left(\frac{1}{2\xi}\right)\right.\\ &\left. -\frac{1}{\xi}\ln\left(\frac{1}{2\xi}\right) + \frac{2}{3}\ln\left(\frac{1}{2\xi}\right) - \frac{1}{\xi}\ln 2\pi\right],\end{aligned}$$

$$\xi = B/B_c$$

$$\begin{aligned}\mathcal{L}_{GG} &= \frac{\alpha}{2\pi B_c^2\xi^2}\left[-\frac{1}{3} - \frac{2}{3}\psi\left(1 + \frac{1}{2\xi}\right) - \frac{1}{2\xi^2} + \frac{2}{3}\xi + \right.\\ &\left. 8\zeta'\left(-1, \frac{1}{2\xi}\right) - \frac{1}{2\xi}\ln\Gamma\left(\frac{1}{2\xi}\right) + \frac{1}{\xi}\ln 2\pi - \frac{1}{\xi}\ln\left(\frac{1}{2\xi}\right)\right]\end{aligned}$$

- L_{GG} has a divergent term that grows like ξ

*: J. Lundin, Europhys. Lett. 87, 31001 (2009)

EULER-HEISENBERG DIPOLE

- As we said, we use as background the magnetic field of a dipole oriented along the z axis

$$\phi_0(\mathbf{r}) = \sqrt{\frac{4\pi}{3}}\frac{m}{r^2}Y_{10}(\theta,\phi) \qquad \mathbf{B}_0 = -\boldsymbol{\nabla}\phi_0$$

- Being the former expression of order α, the zero order solution for the field is sufficient to obtain the first order correction

- Introducing the parameter

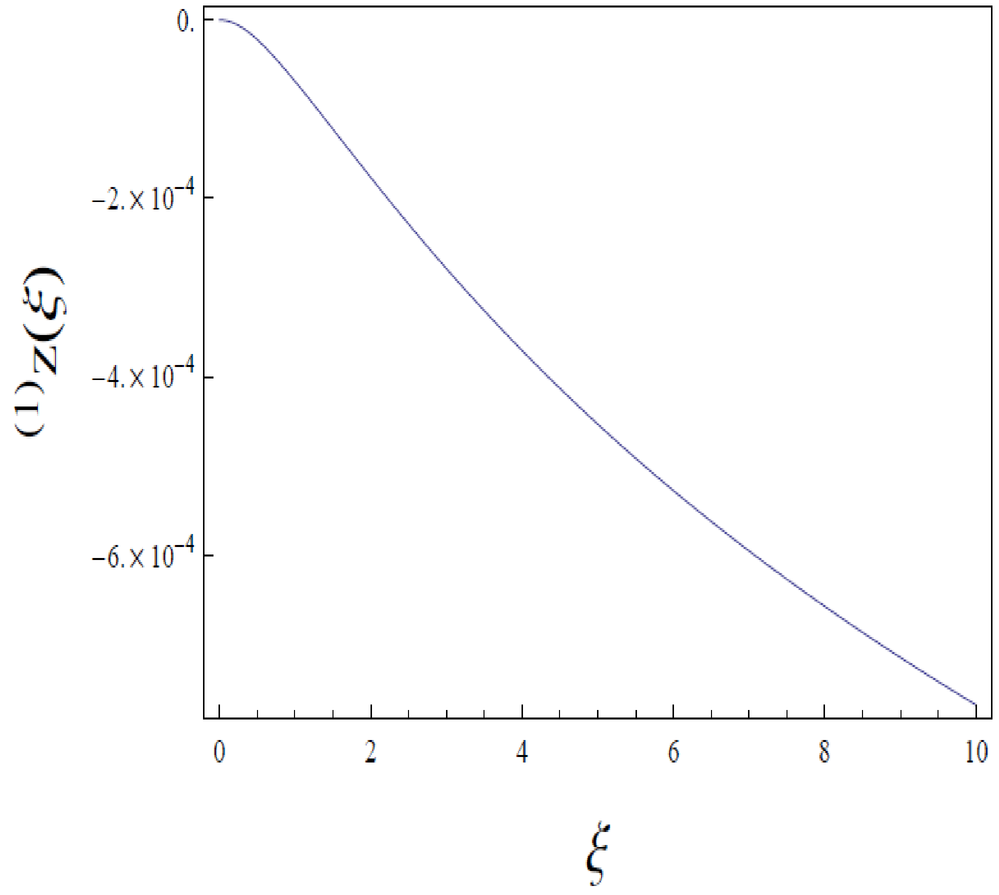
$$\lambda = \frac{m}{r_E^3 B_c}$$

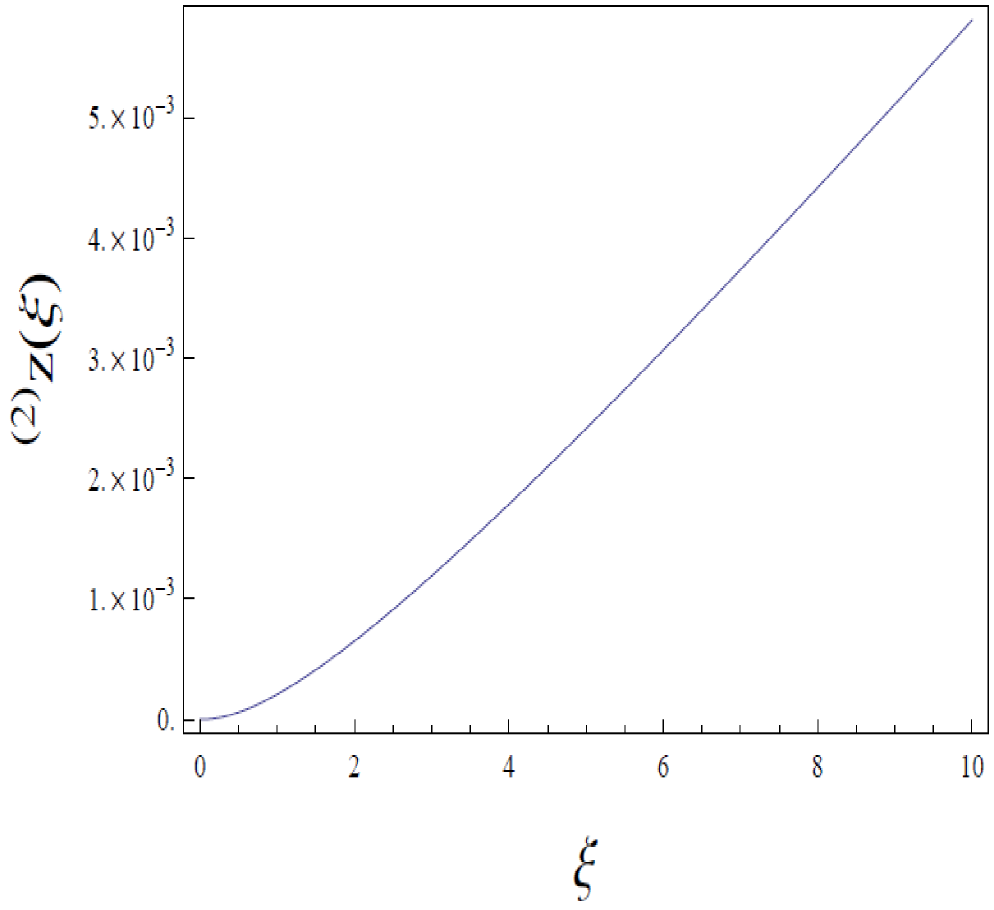
we have that

$$\xi = \frac{B_0}{B_c} = \lambda(1 + 3\cos^2\theta)$$

EULER-HEISENBERG DIPOLE

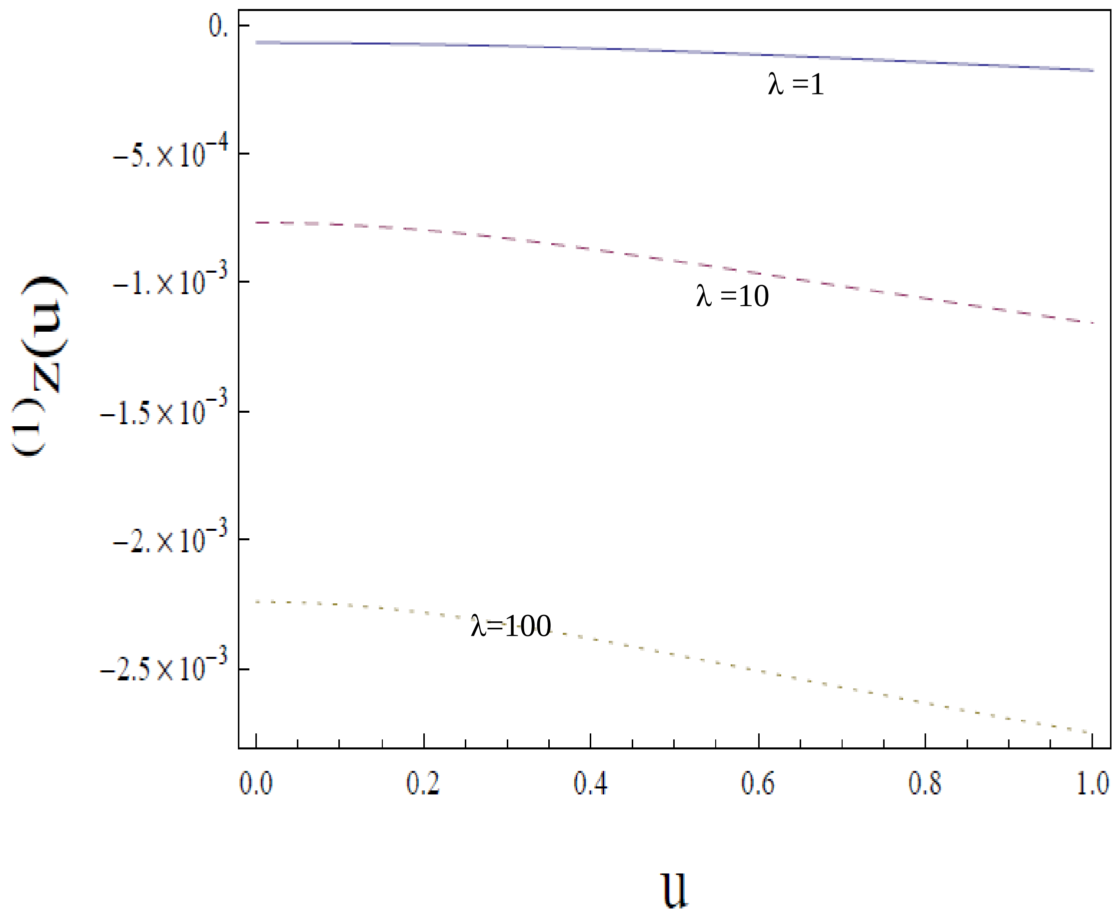
RESULTS

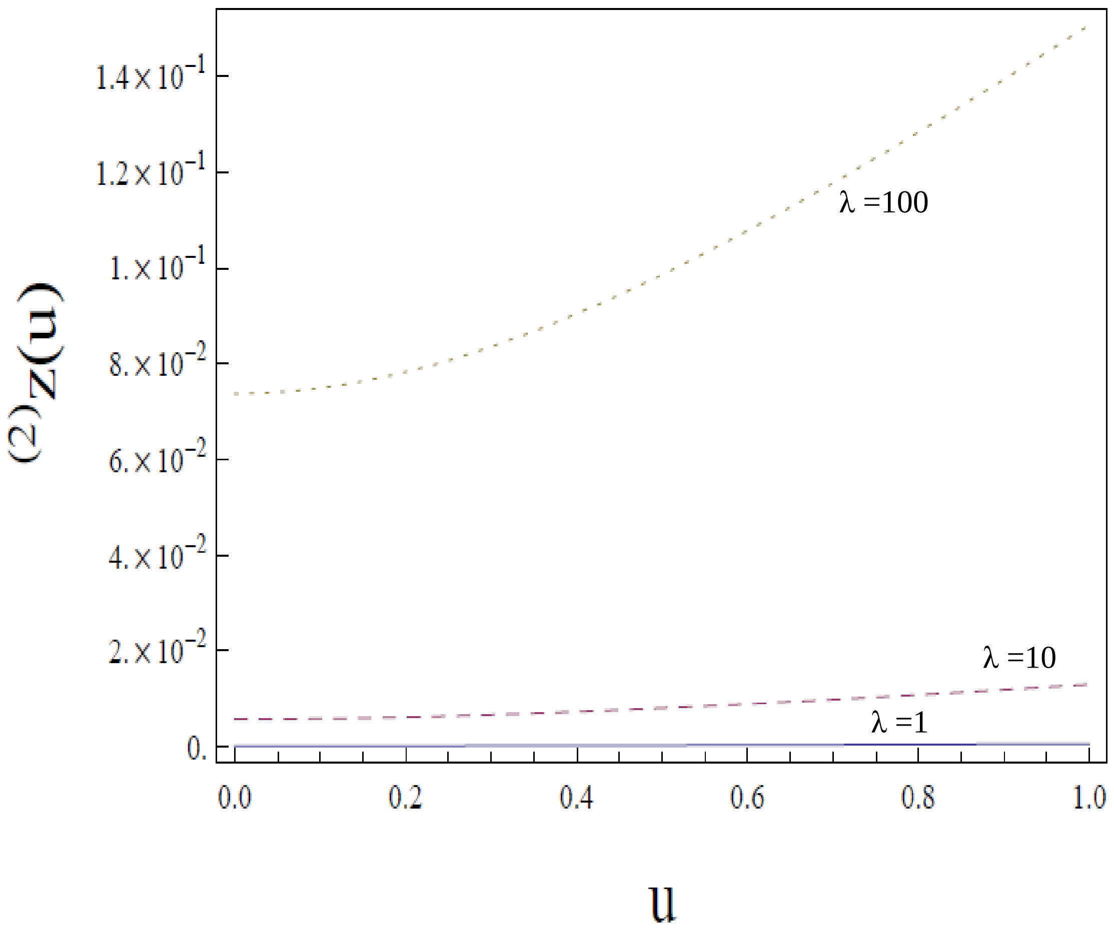




EULER-HEISENBERG DIPOLE

RESULTS





EULER-HEISENBERG DIPOLE

RESULTS

- The shift is small for both the metrics. But for the first it's positive and hence a redshift. For the second is negative, therefore it's a blueshift.

- The second grows faster due to the divergent ξ term in $L_{GG.}$

- The plots show the values of the shift as function of θ for different values of λ that might be significant in the case we wanted to approximate a magnetar with a simple dipole. In fact, we can imagine the photon being emitted at the magnetar surface where the field is up to 100 times the critical one.

- The shift grows in absolute value for both polarizations. But for the first one it has a maximum at the equator whereas this latter is a minimum for the second polarization.

- For a star of 2 solar masses and a radius of 10 km, the gravitational redshift would be 0,2. Hence we can see that our result is comparable to the second polarization shift and a couple order of magnitude bigger than the one given by the first metric.

- We expect the effect to be increased adding rotation to the dipole.

- These results have been submitted to CQG and presented in the MGM.

ACCELERATING PARTICLE IN BORN-INFELD

- In Maxwell’s theory we know what the EM field of an accelerating pointlike charge is. But we also know that this field is singular at the source.

- We would like to obtain this field in the Born-Infeld theory, because we know that in this theory the singularity is smoothed.

- Unfortunately, due to its non linearity, radiating solution of BI theories are not known.

- Our purpose is then to develop new methods to tackle this problem.

ACCELERATING PARTICLE IN BORN-INFELD

PROBLEMS IN CALCULATING THE POTENTIAL IN NL THEORIES

- How is the field of a relativistic charged particle that moves with 4-velocity $V^\alpha(\tau)$ computed?

One computes the Liènard-Wiechert potentials

$$A^\alpha(x) = \left.\frac{eV^\alpha(\tau)}{V\cdot[x-r(\tau)]}\right|_{\tau=\tau_0}$$

and from these obtains the fields by differentiation.

But these potentials are computed through the Green functions $D_r(x-x')$

$$A^\alpha(x) \quad = \quad 4\pi\int d^4x' D_r(x-x')\, J^\alpha(x')$$

ACCELERATING PARTICLE IN BORN-INFELD

PROBLEMS IN CALCULATING THE POTENTIAL IN NL THEORIES

- By definition, Green functions are the impulse response of an inhomogeneous differential operator on a certain domain, once we specified the boundary conditions.

- Via superposition, the convolution of the Green function with an arbitrary function gives the response of the inhomogeneous operator to that function.

- **If we lose linearity, we cannot exploit this property!**

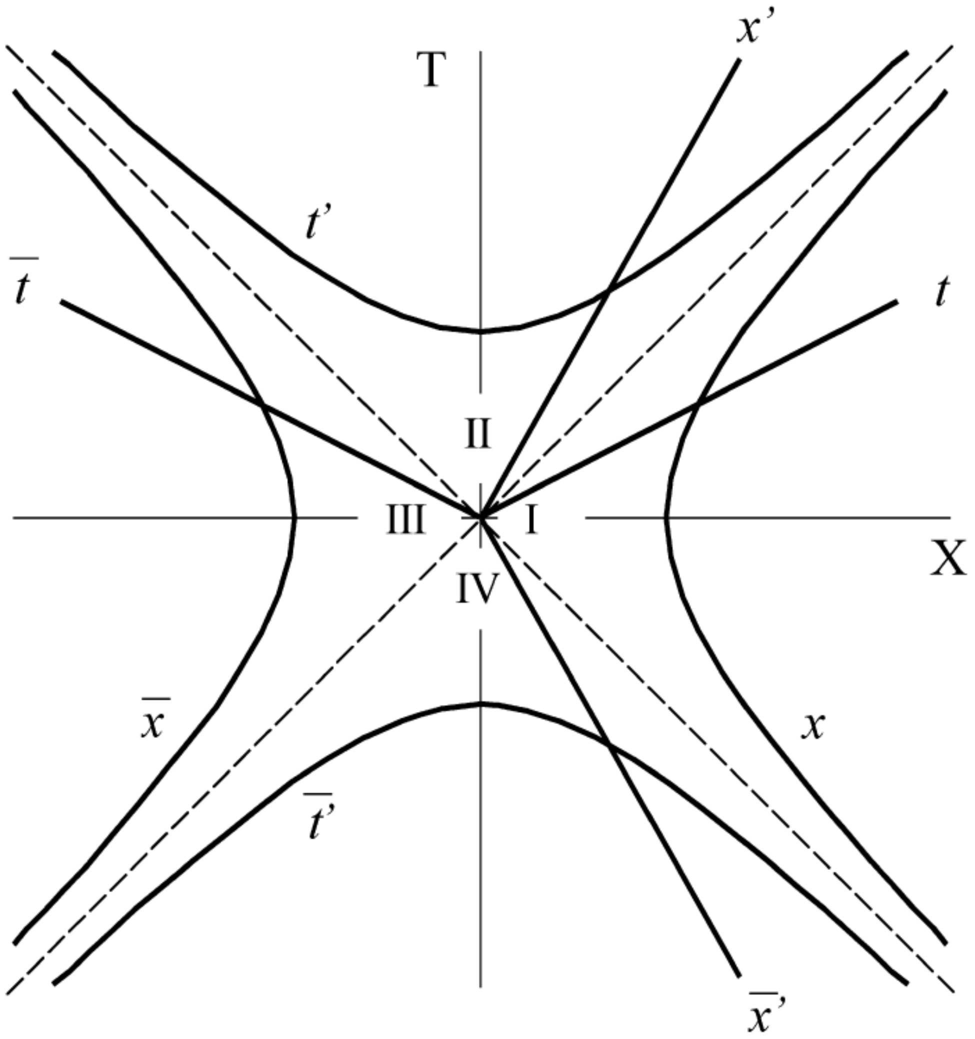
- The problem becomes even more difficult if the particle is accelerating.

ACCELERATING PARTICLE IN BORN-INFELD

RINDLER SPACETIME

- A different route to solve the problem is being investigated by us.
- Instead of computing the potentials and fields in the inertial reference frame where the particle moves, one might compute them in a **non-inertial, comoving** frame where the particle is at rest.
- Once one has those fields, it suffices to transform them back to the inertial frame.
- The most apt coordinate tranformation to describe a relativisic accelerating particle is that of Rindler.

ACCELERATING PARTICLE IN BORN-INFELD



Rindler, quadrant I $X^2 > T^2, X > 0$	$gt = ar\tanh\frac{T}{X}$	$x = \sqrt{X^2 - T^2}$	$-\infty < t < \infty$ $0 \leqslant x < \infty$
Inverse	$T = x \sinh gt$	$X = x \cosh gt$	$-\infty < T < \infty$ $0 \leqslant X < \infty, X > \lvert T \rvert$
Milne, quadrant II $X^2 < T^2, T > 0$	$t' = \sqrt{T^2 - X^2}$	$x' = ar\tanh\frac{X}{T}$	$0 \leqslant t' < \infty$ $-\infty < x' < \infty$
Inverse	$T = t' \cosh x'$	$X = t' \sinh x'$	$0 \leqslant T < \infty, T > \lvert X \rvert$ $-\infty < X < \infty$
Anti Rindler, quadrant III $X^2 > T^2, X < 0$	$g\bar{t} = -ar\tanh\frac{T}{X}$	$\bar{x} = -\sqrt{X^2 - T^2}$	$-\infty < \bar{t} < \infty$ $-\infty < \bar{x} \leqslant 0$
Inverse	$T = -\bar{x} \sinh g\bar{t}$	$X = \bar{x} \cosh g\bar{t}$	$-\infty < T < \infty$ $-\infty < X \leqslant 0, X < -\lvert T \rvert$
Anti Milne, quadrant IV $X^2 < T^2, T < 0$	$\bar{t}' = -\sqrt{T^2 - X^2}$	$\bar{x}' = -ar\tanh\frac{X}{T}$	$-\infty < \bar{t}' \leqslant 0$ $-\infty < \bar{x}' < \infty$
Inverse	$T = \bar{t}', \cosh \bar{x}'$	$X = -\bar{t}' \sinh \bar{x}'$	$-\infty < T \leqslant 0, T < -\lvert X \rvert$ $-\infty < X < \infty$

ACCELERATING PARTICLE IN BORN-INFELD

RINDLER SPACETIME

- The price to pay for having a simple equation is that now the coordinates are curved, to keep track of tha fact that, originally, the frame was non inertial.

- In their covariant form, valid for every reference frame, Maxwell's equations are

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu})=4\pi j^\nu$$

- Rindler's metric can be written in many forms. Chosing

$$ds^2 = (1+gx)^2\, d\tau - dx^2 - dy^2 - dz^2$$

the solution is

$$E_x = 4qx_0^2\frac{x^2-x_0^2-\rho^2}{\xi_R^3} \qquad \rho = \sqrt{y^2+z^2}$$

$$E_\rho = 8qx_0^2\frac{\rho x}{\xi_R^3} \qquad \zeta_{\mathrm{R}} = [(x^2+\rho^2-x_1^2)^2+4x_1^2\rho^2]^{1/2} = [(x^2+\rho^2+x_1^2)^2-4x_1^2x^2]^{1/2}$$

ACCELERATING PARTICLE IN BORN-INFELD

RINDLER SPACETIME

- Now one simply has to transform back to the iniertial frame.

- In principle this method can be used also in the case of NLEM theories, in fact every reference to Green functions has been eliminated.

- However there are some more issues to solve. If one is set to solve the differential equation in the NL context, one has the equations of motions given by

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\cdot\frac{F^{\mu\nu}}{\sqrt{1+2F}}\right]=j^\nu$$

Of course the appearance of the field in the denominator – especially inside a square root – complicates the equation a lot and makes the method mathematically difficult to apply.

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- We start from Maxwell's equation in the Rindler spacetime. The one relevant to us read

$$\vec{\nabla} \times \left[(1 + gx) \vec{E} \right] = 0 \qquad E_i = (1 + gx) F^{0i}$$

$$\vec{\nabla} \cdot \vec{E} = \rho$$

where g is the particle acceleration.

- So the electric potential is defined by

$$(1 + gx) \vec{E} = \vec{\nabla}\Phi \quad \Longrightarrow \quad \vec{E} = \frac{\vec{\nabla}\Phi}{(1 + gx)}$$

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- The Born-Infeld constitutive relation is*

$$\vec{D} = \frac{\vec{E}}{\sqrt{1 - \frac{\left|\vec{E}\right|^2}{b^2}}}$$

which can be expressed in terms of the potential

$$\vec{D} = \frac{\frac{\vec{\nabla}\Phi}{(1+gx)}}{\sqrt{1 - \frac{\left|\frac{\vec{\nabla}\Phi}{(1+gx)}\right|^2}{b^2}}} = \frac{\vec{\nabla}\Phi}{(1+gx)} \cdot \frac{1}{\sqrt{1 - \frac{\nabla\Phi^2}{b^2(1+gx)^2}}}$$

*: Ferraro R., 2013, J. High Energy Phys., Vol. 2013, 8

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- Because of Born-Infeld's theory peculiar structure, we can use the equation for the divergence of the electric field, provided that we use the field D instead of E. So, the dynamical equation for our field is given by

$$\vec{\nabla} \cdot \left[-\frac{\vec{\nabla}\Phi}{(1+gx)} \cdot \frac{1}{\sqrt{1 - \frac{\nabla\Phi^2}{b^2(1+gx)}}} \right] = q\delta\left(\vec{x} - \vec{x}_0\right)$$

which a minus sign appeared in the left had side because we consider a point charge and, for simplicity, we have chosen the electron, but taking the charge value q positive.

- Φ is the full Born-Infeld field, which is composed by a Maxwellian part plus a small correction

$$\Phi = \Phi_0 + \varphi$$

$$\Phi_0 \gg \varphi$$

$$\frac{\nabla\Phi^2}{b^2} \ll 1$$

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- Using the binomial series

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

we can expand the square root at the denominator in our field equation, getting

$$\vec{\nabla} \cdot \left[-\frac{\vec{\nabla}\Phi}{(1+gx)} \cdot \left(1 + \frac{1}{2} \frac{\nabla\Phi^2}{b^2 (1+gx)^2} \right) \right] = q\delta\left(\vec{x} - \vec{x}_0\right)$$

- We don't need to worry about convergence because we have that the binomial series is convergent for

1. $|x| < 1$ and $\forall \alpha \in \mathbb{C}$
2. $x = -1 \iff Re\left(\alpha\right) > 0 \; or \; \alpha = 0$
3. $|x| = 1 \wedge \; x \neq -1 \Rightarrow Re\left(\alpha\right) > -1$
4. $|x| > 1$ and α is negative.

$\Rightarrow$ In our case, α<0 and |x|≠1

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- Expanding the field Φ in the first term in its Mawellian component plus the correction, one gets

$$\vec{\nabla}\cdot\left[-\frac{\vec{\nabla}\Phi_0}{(1+gx)}-\frac{\vec{\nabla}\varphi}{(1+gx)}-\frac{1}{2}\frac{\vec{\nabla}\Phi\ \nabla\Phi^2}{b^2\,(1+gx)^3}\right]=q\delta\left(\vec{x}-\vec{x}_0\right)$$

- From this expression we can see that, formally we have an operator $\hat{O}$ acting on the field, where $\hat{O}$ is composed – just like the field – by a Maxwellian part plus a correction. In other terms

$$\left(\hat{O}_M+\hat{O}'\right)(\Phi_0+\varphi)=-q\delta$$

- By definition

$$\hat{O}_M\Phi_0=-q\delta$$

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- So

$$\hat{O}_M \Phi_0 + \hat{O}_M \varphi + \hat{O}' \Phi_0 + \hat{O}' \varphi = -q\delta$$
$$\hat{O}_M \varphi + \hat{O}' \Phi_0 + \hat{O}' \varphi = 0$$

yielding, for our equation

$$\vec{\nabla} \cdot \left[-\frac{\vec{\nabla}\varphi}{(1+gx)} - \frac{1}{2} \frac{\vec{\nabla}\Phi \; \nabla\Phi^2}{b^2 \, (1+gx)^3} \right] = 0$$

- Expanding the remaining Φ's

$$\vec{\nabla} \cdot \left[-\frac{\vec{\nabla}\varphi}{(1+gx)} - \frac{1}{2} \frac{\nabla\Phi_0^2 \; \vec{\nabla}\Phi_0 + 2\left(\vec{\nabla}\Phi_0 \cdot \vec{\nabla}\varphi\right) \vec{\nabla}\Phi_0 + \nabla\Phi_0^2 \; \vec{\nabla}\varphi}{b^2 \, (1+gx)^3} \right] = 0$$

Where terms of higher order in φ have been neglected

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- Because the Mawell field is much bigger than the correction the leading term in the last expansion is the one where only Φ_0 appears; hence

$$\vec{\nabla} \cdot \left[-\frac{\vec{\nabla}\varphi}{(1+gx)} - \frac{1}{2}\frac{\nabla\Phi_0^2 \; \vec{\nabla}\Phi_0}{b^2\,(1+gx)^3} \right] = 0$$

from which

$$\frac{g}{(1+gx)}\nabla^2\varphi - \frac{g^2}{(1+gx)^2}\partial_x\varphi = g \cdot \vec{\nabla} \cdot \left[-\frac{1}{2}\frac{\nabla\Phi_0^2 \; \vec{\nabla}\Phi_0}{b^2\,(1+gx)^3} \right]$$

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- Now we perform the change of variable

$$\xi = \frac{1 + gx}{g}$$

which preserves the measure $$dx = d\xi$$

- We obtain the final equation

$$\frac{\nabla^2 \varphi}{\xi} - \frac{\partial_x \varphi}{\xi^2} = \frac{1}{g^2} \cdot \vec{\nabla} \cdot \left[-\frac{1}{2} \frac{\nabla \Phi_0^2 \, \vec{\nabla} \Phi_0}{b^2 \xi^3} \right]$$

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- The equation governing φ is wave equation, which explicitly exhibits the axial symmetry of the problem and with a source that, unlike what happens in the common case, it isn't localized on a specific wordline. Instead it is a diffused field that cover the entire spacetime.

- The memory of the particle position is carried because the source depends on Φ_0, which depends on the workdline.

- To solve this equation, the most straightforward method is – as usual – to compute the Green function and then covolute it with the source.

$$\frac{\nabla^2 G}{\xi} - \frac{\partial_x G}{\xi^2} = \delta\left(\xi - \xi_0\right)\delta\left(y - y_0\right)\delta\left(z - z_0\right)$$

- This procedure is now sound because the theory has been linearized

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- Luckily, the Green function for such an equation has already been computed. In fact, in [1]m the authors show that, for an accelerating Mawellian point particle, described in Rindler's spacetime, the wave equation reads

$$\frac{\nabla^2\Psi}{1+gx} - \frac{\partial_x\Psi}{(1+gx)^2} = -4\pi\delta\left(x-x_0\right)\delta\left(y-y_0\right)\delta\left(z-z_0\right)$$

which is basically the one we need, but for some numerical factors.

- This reference allows us to take

$$G\left(\vec{X},\vec{X}_0\right) = \frac{\xi^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2+\xi_0^2}{4\pi\sqrt{\left[\xi^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2+\xi_0^2\right]^2-4\xi^2\xi_0^2}}$$

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- For the same reason

$$\Phi_0 = \frac{q\left[\xi^2 + (y-y_0)^2 + (z-z_0)^2 + \xi_0^2\right]}{\sqrt{\left[\xi^2 + (y-y_0)^2 + (z-z_0)^2 + \xi_0^2\right]^2 - 4\xi^2\xi_0^2}}$$

- From this we can compute the source which results to be

$$S\left(\vec{X}, \vec{X}_0\right) = -\frac{256q^3}{b^2g^2} \cdot \frac{\xi\ \xi_0^6}{\left\{\left[\xi^2 + (y-y_0)^2 + (z-z_0)^2 + \xi_0^2\right]^2 - 4\xi^2\xi_0^2\right\}^{\frac{7}{2}}}$$

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

- Therefore, the integral we need to compute is

$$I\left(\vec{X}\right) = -\frac{256q^3}{4\pi b^2 g^2}\int\left[\frac{\xi^2+(y-b)^2+(z-c)^2+a^2}{\sqrt{\left[\xi^2+(y-b)^2+(z-c)^2+a^2\right]^2-4\xi^2a^2}}\cdot\right.$$

$$\left.\frac{a\ \xi_0^6}{\left\{\left[a^2+(b-y_0)^2+(c-z_0)^2+\xi_0^2\right]^2-4a^2\xi_0^2\right\}^{\frac{7}{2}}}\right]dadbdc$$

- Unfortunately, we weren’t able to find any analytical methods to perform the integration.

ACCELERATING PARTICLE IN BORN-INFELD

APPROACH

$$I\left(\vec{X}\right) = -\frac{256 q^3}{4\pi b^2 g^2}\int\left[\frac{\xi^2 + (y-b)^2 + (z-c)^2 + a^2}{\sqrt{\left[\xi^2 + (y-b)^2 + (z-c)^2 + a^2\right]^2 - 4\xi^2 a^2}}\cdot\right.$$

$$\left.\frac{a\ \xi_0^6}{\left\{\left[a^2 + (b-y_0)^2 + (c-z_0)^2 + \xi_0^2\right]^2 - 4a^2\xi_0^2\right\}^{\frac{7}{2}}}\right] dadbdc$$

- The main reason why the integral is difficult to compute is that, like we have pointed out before, the source is a field. Hence, instead of the usual Green function divergence, we have two divergencies.

- We are now trying to compute the result numerically. In particular, we want to expand each denominator in series and compute, term by term, the integral of the product. This should be equivalent to performing a purely numerical integration, but shortens the running time of the program.

- To implement our method we must find the numerical infinity, that is, to find when the result for two consecutive values of the integration window aren’t indistinguishable within the numerical accuracy.

- Last, but not least we need to prove the convergence, that is the we can indeed cut the series at a certain order because the next one is negligible.

CONCLUSIONS

- The context of non-Maxwellian theories is very lively and interesting

- We have presented a couple of applications that might have some physical relevance and that might give us means to check, in the astrophysical context, if non linear theories are nor valid.

- We have shown a possible application of EH formalism to the case of a overcritical magnetic dipole, which is a toy model toward a more consistent model of a magnetar.

- In this framework the results obtained (submitted to CQG and presented in MGM) were comparable or two orders of magnitude smaller with respect to the gravitational redshift as expected. In both cases the shift, in absolute value grows in going from the equator to the poles. But one is a redshift, the other a blueshift.

- We have tackled the problem of computing the field for an accelerating particle in the Born-Infeld theory, underlining the difficulties that arise because of the non linearities

- Through linearization at first order of the theory we have come up with a setup that reduces the problem to a purely computational task.

THANK YOU FOR YOUR ATTENTION